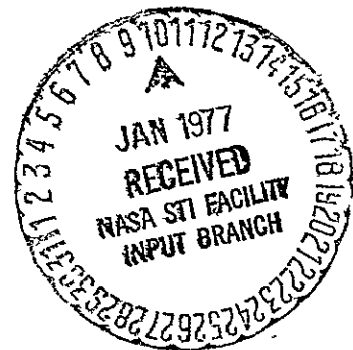NASA TM X-71243

TEST PARTICLE PROPAGATION IN MAGNETOSTATIC TURBULENCE II. THE LOCAL APPROXIMATION METHOD

A. J. KLIMAS
G. SANDRI
J. D. SCUDDER
D. R. HOWELL

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TEST PARTICLE PROPAGATION IN MAGNETOSTATIC TURBULENCE

II. THE LOCAL APPROXIMATION METHOD

by

A. J. Klimas
G. Sandri*
J. D. Scudder
D. R. Howell

Laboratory for Extraterrestrial Physics
Goddard Space Flight Center
Greenbelt, Maryland

*A R A P , Inc , 50 Washington Rd , Princeton, N J , 08540

TEST PARTICLE PROPAGATION IN MAGNETOSTATIC TURBULENCE

II. THE LOCAL APPROXIMATION METHOD

ABSTRACT

A new approximation method for statistical mechanics is presented and applied to a class of problems which contains the test particle propagation problem of paper I. All of the available basic equations used in statistical mechanics are cast in the form of a single equation which is integrodifferential in time and which is then used as the starting point for the construction of the "local approximation method". Simplification of the integrodifferential equation is achieved through approximation to the Laplace transform of its kernel. The approximation is valid near the origin in the Laplace space and is based on the assumption of small Laplace variable. No other small parameter is necessary for the construction of this approximation method. The n'th level of approximation is constructed formally and the first five levels of approximation are calculated explicitly. It is shown that each level of approximation is governed by an inhomogeneous partial differential equation in time with time independent operator coefficients. The order in time of these partial differential equations is found to increase as n does. At n = 0 the "most local" first order partial differential equation which governs the Markovian limit is regained. As n reaches infinity the original integrodifferential equation which is non-local in

time is recovered. The gap between the Markovian limit and the non-local integrodifferential equation is found spanned, with increasing n, by a system of increasingly non-local theories. The local approximation method is applied to two examples. The first of these is a simple exactly soluble case which is obtained through the choice of an appropriate C-number kernel in the integro-differential equation. The exact solution and all of its local approximations are compared for this case. The second example is the quasi-linear diabatic equation of paper I. which governs the probability distribution function for test particles in a simple model of magnetostatic turbulence. In this case the first $(n = 1)$ local approximation is evaluated numerically and presented for the same initial conditions which were used in paper I. For both examples, with the exception of the Markovian limit, the local approximations generally produce very accurate approximations to the exact solutions whenever they evolve slowly in time, or they give time averages of the exact solutions whenever they evolve rapidly. In particular, the failure demonstrated in paper I. of the Fokker-Planck (velocity space diffusion) equation which governs the test particle probability distribution function in the Markovian limit is removed by the "almost Markovian" first local approximation.

TEST PARTICLE PROPAGATION IN MAGNETOSTATIC TURBULENCE

II. THE LOCAL APPROXIMATION METHOD

I. INTRODUCTION

In the first paper in this series[1] we have seen that the familiar Markovian characterization of the propagation of test particles in a simple model of magnetostatic turbulence fails to accurately represent the quasi-linear approximation to the probability distribution function. In our effort to replace the adiabatic approximation, which leads to the Markovian picture, we have discovered a new "local approximation method". In this paper we develop the local approximation method as it applies to a class of integrodifferential equations which contains the quasi-linear diabatic equation that we constructed in paper I. as well as all of the fundamental equations which govern other particle and fluid systems at the kinetic and transport levels.

We outline the local approximation method in general terms, construct the n'th level of approximation formally, and calculate the first five levels of approximation explicitly. We show that each level of approximation is governed by an inhomogeneous partial differential equation in time with time independent operator coefficients. At n = 0 the adiabatic approximation is recovered. Because the adiabatic approximation is governed by a first order partial differential equation in time, its evolution at any instant depends only on its state at

that same instant. This first order equation gives the differential analogue of the Markov chain in which the state of a physical system is imagined to proceed in distinct steps according to a transition probability which is independent of time. In the Markov chain the state of the system after a step forward in time can be given entirely in terms of the state of the system at the preceding step. In the Markovian picture, the adiabatic approximation, the solution of the system under consideration is governed by "local" equations; the evolution of the system at any instant does not depend on the state of the system at previous times.

As n is increased the order in time of the partial differential equation which generates the n'th local approximation also increases. At $n = \infty$, the local approximation is governed by the infinite order partial differential equation which is equivalent to the original integrodifferential equation; i.e., the exact solution is regained as n goes to infinity. This integrodifferential equation is non-local in time; the evolution of its solution at any instant depends on the state of its solution for all previous times.

Within the local approximation method, the zero'th local (or adiabatic) approximation produces the "most local" Markovian description of the solution under consideration. The other extreme ($n = \infty$) gives the exact solution which is governed by the integrodifferential equation which is non-local in time. Between these two extremes we have an infinite set of partial differential equations in time with members whose order increases with increasing n, and

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therefore, whose degree of "localization" decreases with increasing n. The local approximation method provides a mechanism for bridging the large gap between the completely Markovian theory and the completely non-Markovian theory with a system of increasingly non-local theories.

The construction of the local approximation method requires no small parameter. Consequently, we feel that this method will provide many important generalizations of presently available kinetic theories into parameter regimes which have not been theoretically accessible previously. Simplifications of the original intergrodifferential equation are produced by constructing its Laplace transform and then approximating the Laplace transform of its kernel near the origin; i.e., it is the small Laplace variable which is used to produce the approximation. Padé approximants[2] to the Laplace transform of the kernel are used to insure that the approximation which is constructed for small Laplace variable does not fail catastrophically for large values of the Laplace variable. Our application of this method is limited in this paper to those approximants which lie near the diagonal of the Padé table and are equivalent to continued fraction convergents.[5] The order of the convergent for the kernel transform determines the level of the local approximation to the solution. We show that within the class of integrodifferential equations which we consider, there is an identifiable sub-class which contains members equivalent to finite order partial differential equations in time with time independent operator coefficients. This

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sub-class is identified by those kernel transforms which can be expressed as the operator ratio of two finite order polynomials in the Laplace variable.[6] In this sub-class the local approximation method converges to the exact solution for finite n. All higher order local approximations are equivalent, and produce the exact solution.

The source terms in the inhomogeneous equations which generate the local approximations are zero for all time except at the origin in time where they are singular. Thus, the solutions of these equations are double valued at the origin in time; they jump from the true initial conditions to "reinitialized" conditions. Each local approximation has its own unique reinitialization. At least in the examples which we will present in this paper, we often find rapid transients in the exact solutions which are short lived and which lead to slow evolution for larger times. The low order local approximations which we investigate explicitly in this paper are not capable of supporting these rapid transients. Nevertheless, reinitialization seems to choose the appropriate particular solution so that once the transients are over the local approximations and the exact solution agree very well. During the transients the local approximations give an indication of the time average behavior of the exact solution. Thus, with the local approximation method we are able to study the initial value problem without the necessity of computing the details of the often intricate initial transients.

We apply the local approximation method to specific problems and present the results. In section III, we consider an exactly soluble example of the subclass of integrodifferential equations discussed above. A simple kernel function is chosen which leads to the reduction of the original integrodifferential equation to an ordinary third order differential equation with constant C-number coefficients. Thus, the exact solution is readily available. The system of local approximations truncates at n = 4 in this case, and produces a series of first, second and one third order (the exact) ordinary differential equations with constant C-number coefficients. Thus, each of the local approximations is readily available as well. The various local approximations are presented and compared to the exact solution. The details of this comparison are presented in section III. Generally speaking, we find that when the local approximation method produces meaningful results they either take the form of very accurate approximations to the exact solution whenever it evolves slowly in time, or they give a time-averaged representation of the exact solution if it evolves rapidly.

In section IV, we apply the local approximation method to the test particle propagation problem. With our choice of exponential correlation function given in paper I., the system of local approximations to the quasi-linear diabatic probability distribution function truncates at n = 4. In paper I. we have presented several numerical evaluations of the exact quasi-linear diabatic solutions as well as the adiabatic approximation to one of these solutions for an important

initial condition. In the terminology of this paper we have presented several numerical evaluations of the fourth (exact) and the zero'th (adiabatic) local approximations to the quasi-linear diabatic probability distribution function in paper I.. In this paper we present numerical evaluations of the first local approximation to the quasi-linear diabatic solutions for the same initial conditions which were considered in paper I.

In terms of numerical integration, we find that the local approximation represents a considerable simplification. Various numerical instabilities which appear in the quasi-linear diabatic numerical evaluations never appear in the approximate numerical evaluations, and a factor of 10^2 to 10^3 less computer time is found necessary to achieve the same point in time in the approximate solutions compared to the exact ones.

We show by direct comparison that following the initial transients in the exact solutions, which last somewhat less than one Larmor period, the first local approximation becomes very accurate for each of the initial conditions considered. This accuracy persists through the ends of each of the numerical evaluations. Some indication of the accuracy of the first local approximation for time beyond the end of our numerical evaluations is obtained by a study of the Laplace model expansion of the probability distribution function which was introduced in paper I. We show that the first local approximation to some of the low order Laplace modes is very accurate for small values of the Laplace

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variable. We also show that the eigenvalues are accurately given by the first local approximation. These results indicate that the accuracy of the first local approximation persists. In the next paper in this series we develop this reasoning much further in order to obtain strong evidence that the first local approximation remains accurate for all post transient times.

II. CONSTRUCTION OF THE LOCAL APPROXIMATION METHOD

We present the local approximation method through its application to the equation,

$$\frac{\partial f(\tau)}{\partial \tau} = \epsilon \int_0^\tau d\lambda \, K(\lambda) \, f(\tau - \lambda) \qquad (II.1)$$

in which the kernel, $K(\lambda)$, is a linear operator which we will assume has a Laplace transform with a Maclaurin series expansion in the Laplace variable. Other variables (e.g., phase space variables) will be assumed in both $K(\lambda)$ and $f(\tau)$ but will not be exhibited explicitly here. The quasi-linear diabatic equation for test particle propagation in magnetostatic turbulence which was constructed in paper I. is of this form, and in addition, all of the basic equations used in the field of statistical mechanics and some important macro-turbulence theories fall in this classification as well (see Appendix A).

In equation II.1, ϵ is a parameter which is ordinarily assumed small in order to produce kinetic approximations for the long time, near equilibrium evolution of $f(\tau)$.[7,8] The necessity of this assumption normally limits the available kinetic theories to low density and/or weak interaction systems for which

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the time scale which characterizes the duration of a particle collision is much smaller than the time scale over which $f(\tau)$ evolves. In many situations, even though these time scales may be well separated for typical particle parameters they may become mixed for some more or less important regions in the available particle phase space. In paper I. we have seen an example of mixing of the interaction and evolution time scales in an important region of phase space (near $\mu = 0$) and the consequent failure of the standard adiabatic approximation. In this paper we will see that the local approximation method does not require the clear cut separation of time scales and is thus able to remove the non-uniformity found in paper I. Of further significance, we will find that the construction of the local approximation method does not require ϵ small. Consequently, we feel that the local approximation method will provide many important generalizations of presently available kinetic theories into parameter regimes which have not been theoretically accessible previously.

Equation II.1 can be put in the renewal form,

$$f(\tau) = f(0) + \epsilon \int_0^\tau d\lambda \, D(\lambda) \, f(\tau - \lambda) \qquad \text{II.2}$$

in which,

$$D(\lambda) = \int_0^\lambda ds \, K(s) \qquad \text{II.3}$$

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If $K(\lambda)$ is a C-number kernel, then equation II.2 is a Volterra integral equation on the semi-infinite domain with a specialized form for the kernel (convolution integral under Laplace transformation). Of course, if we are faced with the solution of an equation of this form, we can obtain equation II.1 through differentiation with respect to time whether or not the kernel is a C-number kernel.

Our local approximation method, as we have developed it so far, consists of the following steps; First we put the equation, or system of equations, under consideration into the form of equation II.1. Second, we Laplace transform that equation. This Laplace transformation is possible because the integral term in equation II.1 is a convolution integral under the Laplace transformation (however, see Appendix A where the convolution form is dropped). Third, we construct a Maclaurin series expansion of the Laplace transform of the kernel in powers of the Laplace variable. A basic requirement which we have had to assume so far in our construction of this method is that the Laplace transform of the kernel has a Maclaurin series expansion. Fourth, from the Maclaurin series expansion we construct an operator continued fraction expansion of the kernel. Fifth, we truncate this continued fraction expansion at the n'th order in order to produce the n'th "convergents",[5] and then we substitute the n'th convergent back into the Laplace transform of the original equation in place of the exact Laplace transform of the kernel. At this point we have constructed the equation in Laplace space which generates the n'th local approximation to the Laplace transform of the exact solution. Finally, we inverse Laplace

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transform this equation in order to obtain the equation which generates the n'th local approximation to the exact solution of equation II.1 in τ.

At the end of the procedure outlined above we have constructed a finite order, inhomogeneous, partial differential equation in time with operator coefficients which are constant in time. Except in some special situations which will be discussed below, the order of this partial differential equation can range from first, when $n = 0$, to infinite (the integrodifferential equation which generates the exact solution) when n reaches infinity. The zero'th local approximation, which is generated by a first order partial differential equation in time, is identical to the adiabatic approximation which was discussed in paper I. Thus, the local approximation method produces a series of equations which generate, in the lowest order, the most local, Markovian, approximation to the exact solution, and in the highest order, the exact solution which is generated by the non-local integrodifferential equation; the finite orders in between bridge the gap between these two extremes. By construction, all of these equations generate solutions which have Laplace transforms that accurately approximate the Laplace transform of the exact solution when the Laplace variable is small. We will see in this paper through the various examples which will be presented, that this feature of the local approximations usually results in accurate approximations to the exact solution whenever it evolves slowly, or, whenever it evolves rapidly, to time averages of the exact solution.

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In the remainder of this section we present a detailed construction of the local approximation method as outlined above.

The Laplace transform of equation II.1 is,

$$p\tilde{f}(p) - f(0) = \epsilon\tilde{K}(p)\,\tilde{f}(p) \qquad\qquad \text{II.4}$$

and, in the Laplace space, its exact solution can be written formally as,

$$\tilde{f}(p) = [p - \epsilon\tilde{K}(p)]^{-1}\,f(0) \qquad\qquad \text{II.5}$$

The kernel is assumed to have a Maclaurin series expansion in the Laplace variable;

$$\tilde{K}(p) = \sum_{n=0}^{\infty} \left(\frac{M_n}{n!}\right)(-p)^n \qquad\qquad \text{II.6}$$

in which the M_n are the n'th moments of the kernel, defined through,

$$M_n = \int_0^{\infty} d\lambda\,\lambda^n\,K(\lambda) \qquad\qquad \text{II.7}$$

For the remainder of this paper we adopt the operator ratio convention,

$$\frac{X}{Y} \equiv XY^{-1} \qquad\qquad \text{II.8}$$

With this convention a continued fraction expansion of $\tilde{K}(p)$ can be constructed from the Maclaurin series expansion.

$$\tilde{K}(p) = a_0 + \cfrac{a_1 p}{1 + \cfrac{a_2 p}{1 + \cfrac{a_3 p}{1 + \cdots}}} \qquad\qquad \text{II.9}$$

in which some of the low order coefficients are,

$$a_0 = M_0 \qquad\qquad a_1 = - M_1$$

$$a_2 = \frac{1}{2} M_1^{-1} M_0 \qquad a_3 = \frac{1}{3} M_2^{-1} M_3 - \frac{1}{2} M_1^{-1} M_2 \qquad\qquad \text{II.10}$$

$$a_4 = \left(\frac{1}{3} M_2^{-1} M_3 - \frac{1}{2} M_1^{-1} M_2 \right)^{-1} \left[\frac{1}{12} M_2^{-1} M_4 - \frac{1}{9} (M_2^{-1} M_3)^2 \right]$$

We obtain the n'th convergent, $\tilde{K}_n(p)$, to $\tilde{K}(p)$ when we truncate the continued

fraction expansion in equation II.9 beyond a_n. We obtain the n'th local approxi-

mation to $\tilde{f}(p)$, which we denote, $\tilde{f}_n(p)$, by substituting $\tilde{K}_n(p)$ in equation II.5 for

$\tilde{K}(p)$. Thus, the equation which generates the n'th local approximation to the

Laplace transform of the probability distribution function is,

$$p \tilde{f}_n(p) - f(0) = \epsilon \tilde{K}_n(p) \tilde{f}_n(p) \qquad\qquad \text{II.11}$$

Any kernel transform which can be expressed as a ratio of an l'th order

to an m'th order polynomial in p can then be expanded as in equation II.9 with

all $a_n = 0$ for $n > l + m + 1$.[6] All convergents of higher order than $l + m + 1$

are identical, and are exactly $\tilde{K}(p)$. Thus, if the Laplace transform of the kernel

can be expressed as the operator ratio of two finite order polynomials in p,

then the local approximation method must converge on the exact solution for

finite n.

The (L/M) Pade' approximant[2] to $\tilde{K}(p)$ is given by,

$$(L/M) = \frac{P_L(p)}{Q_M(p)} \qquad\qquad II.12$$

in which $P_L(p)$ is a polynomial in p of degree at most L, and $Q_M(p)$ is a polynomial

in p of degree at most M, and in which the coefficients in P_L and Q_M have been

chosen so that a power series expansion in p of equation II.12 leads to the first

L + M + 1 terms in the Maclaurin series expansion of $\tilde{K}(p)$. A Pade' approximant

exists for each combination of L and M. Thus, an array can be constructed,

called the Pade' Table, in which the element of the array in the L'th row and

M'th column is given by (L/M). A portion of this array is given in figure 1 for

the special example in which $\tilde{K}(p)$ is the ratio of a first to a second order poly-

nomial in p. In this case, all elements of the Pade' Table with L $\geq$ 1 and M $\geq$ 2

are identical and equal to $\tilde{K}(p)$. The M = 0 column contains the L'th order

truncations of the Maclaurin series expansion of $\tilde{K}(p)$; some other elements of

the table are exhibited. The boxes with dark borders contain the n'th convergent

to $\tilde{K}(p)$ for $n = L + M$[5]. For the special example being considered here, all

convergents with n $\geq$ 4 lie in the infinite block of identical approximants given

by $\tilde{K}(p)$. The n'th local approximation to $\tilde{f}(p)$ is obtained by substituting the

(L/M) approximant which has L + M = n, and which lies in the "staircase" of

dark boxes in the Pade' Table, for $\tilde{K}(p)$ in equation II.5.

We have not investigated the consequences of choosing approximants to $\tilde{K}(p)$

other than those which correspond to the convergents. However, we point out

Q-NUMBER PADÉ TABLE

M L	0	1	2
0	M_0	$\dfrac{M_0}{1+M_0^{-1}M_1 P}$	$\dfrac{M_0}{1+M_0^{-1}M_1 P + \left[(M_0^{-1}M_1)^2 - \frac{1}{2}M_0^{-1}M_2\right]P^2}$
1	$M_0 - M_1 P$	$\dfrac{M_0 + \left[\frac{1}{2}M_0 M_1^{-1}M_2 - M_1\right]P}{1+\frac{1}{2}M_1^{-1}M_2 P}$	$\tilde{K}(P)$
2	$M_0 - M_1 P + \frac{1}{2}M_2 P^2$	$(2/1)$	$\tilde{K}(P)$

$$(2/1) = \frac{M_0 + \left[\frac{1}{3}M_0 M_2^{-1}M_3 - M_1\right]P + \left[\frac{1}{2}M_2 - \frac{1}{3}M_1 M_2^{-1}M_3\right]P^2}{1+\frac{1}{3}M_2^{-1}M_3 P}$$

Figure 1. The Padé table for a Q-number kernel whose Laplace transform can be expressed as the ratio of a first to a second order polynomial in the Laplace variable. The operator ratios shown here are to be interpreted according to equation II.8 in the text.

the connection between the convergents and the Padé approximants to indicate a possible generalization of the particular construction which will be presented here. Padé approximants have been used to accomplish approximate Laplace transform inversion,[3] and they have also played an important role as an approximation technique in Hilbert space for various quantum-mechanical models.[4]

The equation which generates the n'th local approximation to $f(\tau)$ can be constructed by inverse Laplace transforming equation II.11. By inverting this equation directly, we find,

$$\frac{\partial f_n(\tau)}{\partial \tau} = \epsilon \int_0^\tau d\lambda \, K_n(\lambda) \, f_n(\tau - \lambda) \qquad \text{II.13}$$

in which the $K_n(\lambda)$ are the inverse Laplace transforms of the convergents for the kernel transform. In Appendix C some of the $K_n(\lambda)$ are computed explicitly. Here we will show that the $f_n(\tau)$ also obey equivalent inhomogeneous partial differential equations in time with time independent operator coefficients. These equations are obtained by first manipulating equation II.11 into a convenient form and then performing the inverse Laplace transform. This procedure is outlined in the following for n = 0 through 4.

a). n = 0

For n = 0, we have,

$$p\tilde{f}_0(p) - f(0) = \epsilon M_0 \tilde{f}_0(p) \qquad \text{II.14}$$

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and therefore,

$$\frac{\partial f_0}{\partial \tau} = \epsilon M_0 f_0 \qquad \text{II.15}$$

with the proviso that $f_0(0) = f(0)$. The formal solution of equation II.15 is,

$$f_0(\tau) = e^{\epsilon M_0 \tau} f(0) \qquad \text{II.16}$$

Equation II.15 can be compared with equations I.26 and I.27 or equation I.36 of paper I. in order to see that the zero'th local approximation ($n = 0$) is just the adiabatic approximation. In this approximation we replace the original integro-differential equation (in time) with a first order differential equation (in time). We have seen in paper I in our study of test particle propagation in magnetostatic turbulence that this level of approximation produces the Markovian description of $f(\tau)$ which is also available from a Fokker-Planck equation in velocity space for $f(\tau)$. In analogy with the Markov chain, we feel that this description of $f(\tau)$ is as local a description of its evolution as can be constructed. We use "local" in contrast to equation II.1 which contains an obvious memory of the past and is clearly non-local in time.

b). $n = 1$.

The first local approximation is generated by,

$$p\tilde{f}_1(p) - f(0) = \epsilon \tilde{K}_1(p)\, \tilde{f}_1(p) = \epsilon(M_0 - pM_1)\, \tilde{f}_1(p) \qquad \text{II.17}$$

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The inverse Laplace transform of this equation is,

$$[1 + \epsilon M_1] \frac{\partial f_1}{\partial \tau} - \epsilon M_0 f_1 = - \epsilon \delta(\tau) M_1 f(0) \qquad \text{II.18}$$

which has a formal solution given by,

$$f_1(\tau) = e^{\epsilon[1 + \epsilon M_1]^{-1} M_0 \tau} [1 + \epsilon M_1]^{-1} f(0) \qquad \text{II.19}$$

Notice that,

$$f_1(\tau) \xrightarrow[\tau \to 0^+]{} [1 + \epsilon M_1]^{-1} f(0) \qquad \text{II.20}$$

Because of the presence of the Dirac delta function on the right hand side of

equation II.18, $f_1(\tau)$ is discontinuous at $\tau = 0$, jumping from the initial condition,

$f(0)$, to the "reinitialized" condition given by equation II.20. Equation II.18 is

also a first order differential equation in time, and in this respect, is similar

to equation II.15. However, the presence of the Dirac delta function enforces a

memory of the origin in time in equation II.18 thereby leading to a non-Markovian

kinetic theory for $f_1(\tau)$. In a sense, this level of approximation is local com-

pared to the integrodifferential equation which generates the exact solution,

but not as local as the Markovian description.

c). n = 2.

The second local approximation is generated by,

$$p\tilde{f}_2(p) - f(0) = \epsilon \tilde{K}_2(p) \tilde{f}_2(p) \qquad \text{II.21}$$

In order to construct the equation which generates $f_2(\tau)$,

$$\tilde{K}_2(p) = a_0 + a_1 p(1 + a_2 p)^{-1} \qquad\qquad \text{II.22}$$

must be first rewritten in an appropriate way.

$$\tilde{K}_2(p) = a_0 + [(1 + a_2 p) a_1^{-1}]^{-1} p$$

$$= a_0 + [a_2 a_1^{-1}(a_1 a_2^{-1} a_1^{-1} + p)]^{-1} p \qquad\qquad \text{II.23}$$

$$= a_0 + (p + a_1 a_2^{-1} a_1^{-1})^{-1} a_1 a_2^{-1} p$$

By substituting this expression into equation II.21 and operating on the equation by $(p + a_1 a_2^{-1} a_1^{-1})$, we obtain,

$$(p + a_1 a_2^{-1} a_1^{-1}) [p\tilde{f}_2(p) - f(0)] = \epsilon [a_1 a_2^{-1} a_1^{-1} a_0 + (a_0 + a_1 a_2^{-1}) p] \tilde{f}_2(p) \qquad \text{II.24}$$

This form can easily be inverse Laplace transformed to

$$\frac{\partial^2 f_2}{\partial \tau^2} + B_2 \frac{\partial f_2}{\partial \tau} + A_2 f_2 = \epsilon \delta(\tau) D_2 f(0) \qquad\qquad \text{II.25}$$

in which,

$$A_2 = - \epsilon a_1 a_2^{-1} a_1^{-1} a_0 \qquad\qquad \text{II.26}$$

$$B_2 = a_1 a_2^{-1} a_1^{-1} - \epsilon(a_0 + a_1 a_2^{-1}) \qquad\qquad \text{II.27}$$

and

$$D_2 = a_0 + a_1 a_2^{-1} \qquad\qquad \text{II.28}$$

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In this case reinitialization leads to

$$f_2(0^+) = f(0)$$

$$\frac{\partial f_2(0^+)}{\partial \tau} = \epsilon D_2 f(0)$$

II.29

Higher order local approximations can be obtained in a manner similar to that used above for the second local approximation. As higher order convergents are considered the operation given by equation II.23 must be repeated an increasing number of times in order, in each case, to reach an equation for $\tilde{f}_n(p)$ which is similar to equation II.24. In this equation, only powers of p multiply either $\tilde{f}_2(p)$ or $f(0)$. Once this form is reached the inverse Laplace transformation can be accomplished easily.

d). n = 3.

We find that the third local approximation is generated by,

$$C_3 \frac{\partial^2 f_3}{\partial \tau^2} + B_3 \frac{\partial f_3}{\partial \tau} + A_3 f_3 = \epsilon \delta(\tau) D_3 f(0) + \epsilon \dot{\delta}(\tau) E_3 f(0)$$

II.30

with,

$$A_3 = - \epsilon a_0 a_1 (1 + a_3 a_2^{-1})^{-1} a_2^{-1} a_1^{-1}$$

II.31

$$B_3 = a_1 (1 + a_3 a_2^{-1})^{-1} a_2^{-1} (a_1^{-1} - \epsilon) - \epsilon a_0$$

II.32

$$C_3 = 1 - \epsilon a_1 (1 + a_2 a_3^{-1})^{-1}$$

II.33

$$D_3 = a_0 + a_1 (1 + a_3 a_2^{-1})^{-1} a_2^{-1}$$

II.34

and,

$$E_3 = a_1 (1 + a_2 a_3^{-1})^{-1} \qquad\qquad \text{II.35}$$

and with reinitialization given by,

$$f_3(0^+) = C_3^{-1} f(0)$$

$$\text{II.36}$$

$$\frac{\partial f_3(0^+)}{\partial \tau} = C_3^{-1} [(D_3 - a_0) a_1^{-1} - B_3 C_3^{-1}]\ f(0)$$

e). $n = 4$.

The fourth local approximation is generated by,

$$\frac{\partial^3 f_4}{\partial \tau^3} + C_4 \frac{\partial^2 f_4}{\partial \tau^2} + B_4\ \frac{\partial f_4}{\partial \tau} + A_4 f_4 = \epsilon\, \delta\, (\tau)\, D_4\, f(0) + \epsilon \dot{\delta}\, (\tau)\, E_4\, f(0) \qquad \text{II.37}$$

with,

$$A_4 = -\epsilon a_1 a_3 a_4^{-1} a_3^{-1} a_2^{-1} a_1^{-1} a_0 \qquad\qquad \text{II.38}$$

$$B_4 = \quad a_1 [a_3 a_4^{-1} a_3^{-1} a_2^{-1} a_1^{-1} - \epsilon (a_3 a_4^{-1} a_2^{-1} a_1^{-1} + a_3 a_4^{-1} a_3^{-1} a_1^{-1} + a_2^{-1} a_1^{-1})\, a_0 - \epsilon a_3 a_4^{-1} a_3^{-1} a_2^{-1}] \quad \text{II.39}$$

$$C_4 = \quad a_1 [a_3 a_4^{-1} a_2^{-1} + a_3 a_4^{-1} a_3^{-1} + a_2^{-1}]\, a_1^{-1} - \epsilon\, [a_0 + a_1 (a_3 a_4^{-1} + 1)\, a_2^{-1}] \qquad \text{II.40}$$

$$D_4 = -\epsilon\, [1 - a_1 (a_3 a_4^{-1} a_2^{-1} + a_3 a_4^{-1} a_3^{-1} + a_2^{-1})\, a_1^{-1} a_0 - a_1 a_3 a_4^{-1} a_3^{-1} a_2^{-1}] \qquad \text{II.41}$$

and

$$E_4 = a_0 + a_1 (1 + a_3 a_4^{-1})\, a_2^{-1} \qquad\qquad \text{II.42}$$

and with reinitialization given by,

$$f_4(0^+) = f(0)$$

$$\frac{\partial f_4(0^+)}{\partial \tau} = \epsilon D_4 f(0) \qquad \text{II.43}$$

$$\frac{\partial^2 f_4(0^+)}{\partial \tau^2} = \epsilon [K(0) - C_4 D_4 - E_4] f(0)$$

The procedure outlined above could be carried out further in order to

obtain even higher order local approximations. In the remainder of this paper,

however, we will apply this method to two examples in which the fourth and

all higher local approximations generate the exact solutions. For our purposes

here, it is not necessary to compute the higher order approximations.

As we proceed down the staircase in the Pade' table with increasing n, or

equivalently, as we adopt higher order convergents for the kernel transform,

we produce higher order local approximations for $f(\tau)$. These local approxi-

mations are generated by partial differential equations of increasing order in

time (the order is given by M + 1) with operator coefficients which are constant

in time, and with reinitializations appropriate for each approximation. If the

kernel transform can be expressed as the ratio of two finite order polynomials

in p, then this procedure truncates at a finite order, and in that order it produces

the partial differential equation which is equivalent to the original integrodif-

ferential equation. Otherwise, this procedure converges on the exact solution,

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which is generated by a true integrodifferential equation, only as n goes to infinity.

All of the partial differential equations which generate the various local approximations have one other feature in common; they all generate solutions which have Laplace transforms that accurately approximate the Laplace transform of the exact solution when the Laplace variable is small. We view this feature as the guiding principle in the construction of the local approximation method, and in this sense, consider the particular construction which we have presented here as only one of many possible such constructions with the same guiding principle.

We have seen that the zero'th local approximation is identical to the adiabatic approximation. At this level of approximation we reproduce the Markovian description of the probability distribution function. In general, at this level the local approximation is governed by a first order differential equation in time which has no reinitialization. This equation is time-translation invariant (it does not contain a memory of the origin in time), and it produces changes of $f_0(\tau)$ at any instant in time which depend only on $f_0(\tau)$ at that instant, very much in the spirit of the Markov chain. We call this equation a local equation in contrast to the non-local integrodifferential equation. The local approximation method, therefore, provides a bridge from the most local Markovian description to the completely non-local integrodifferential description. In paper I. of this series we have seen an example in which the Markovian description fails to

produce an accurate approximation to the exact solution. In section IV. of this paper we will reexamine that case and show that the "almost Markovian" $f_1(\tau)$ provides a very good approximation to the exact solution.

III. APPLICATION TO AN EXACTLY SOLUBLE PROBLEM WITH A C-NUMBER KERNEL

In the event that the kernel under consideration in equation II.1 is a C-number function, the equations which generate the local approximations to $f(\tau)$ reduce to ordinary differential equations with constant coefficients. Thus, their solutions are readily available. In this section we present some examples of these solutions for the C-number kernel,

$$K(\lambda) = e^{-\mu\lambda} \cos \lambda \qquad \qquad \text{II.44}$$

in which μ is a parameter which lies in the domain, $0 \leq \mu \leq 1$. The Laplace transform of this kernel is,

$$\tilde{K}(p) = \frac{p + \mu}{1 + (p + \mu)^2} \qquad \qquad \text{II.45}$$

Since this kernel transform is the ratio of a first to a second order polynomial in p, we expect the fourth, and all higher, local approximations to produce the exact solution. Indeed, with this kernel, and with the use of repeated time differentiation, we can show that equation II.1 is equivalent to,

$$\frac{\partial^3 f}{\partial \tau^3} + 2\mu \frac{\partial^2 f}{\partial \tau^2} + (1 + \mu^2 - \epsilon) \frac{\partial f}{\partial \tau} - \epsilon\mu f = 0 \qquad \qquad \text{II.46}$$

with the additional initial conditions,

$$\frac{\partial f(0)}{\partial \tau} = 0 \qquad \frac{\partial^2 f(0)}{\partial \tau^2} = \epsilon f \qquad\qquad \text{II.47}$$

The kernel transform given in equation II.44 does have a Maclaurin series expansion with some of the lower order moments given by,

$$M_0 = \frac{\mu}{1 + \mu^2} \qquad\qquad M_1 = -\frac{1 - \mu^2}{(1 + \mu^2)^2}$$

$$M_2 = -\frac{2\mu(3 - \mu^2)}{(1 + \mu^2)^3} \qquad M_3 = \frac{6(1 - 6\mu^2 + \mu^4)}{(1 + \mu^2)^4} \qquad \text{II.48}$$

$$M_4 = \frac{24\mu(\mu^4 - 10\mu^2 + 5)}{(1 + \mu^2)^5}$$

Through the substitution of these moments into equation II.10, we are able to show that equations II.37 through II.43, which generate the fourth local approximation, reduce exactly to equation II.46, which generates the exact solution. In this case the entire system of local approximations is comprised of $f_0(\tau)$ through $f_4(\tau)$, and the Pade$'$ table given in figure 1 is directly applicable with the understanding that the moments which appear there are C-numbers. We have chosen to study this particular C-number kernel since it has the same functional dependence on μ and λ as the Q-number kernel which appears in the quasi-linear diabatic equation which was constructed in paper I., and whose first local approximation will be studied in detail in the next section of this paper.

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The local approximations to $f(\tau)$, including the fourth local approximation (the exact solution), are plotted in figures 2 through 5 for $\epsilon = 1.0$ and several values of μ. Notice that in the vicinity of $\mu = 1.0$ the exact solution contains a short-lived weak transient which leads quickly into simple exponential decay. For these values of μ the various local approximations start with individual values and slopes subject to their respective reinitializations, but they all quickly become essentially equivalent and then accurately represent the exact solution once it reaches the post-transient slow exponential decay phase. In the vicinity of $\mu = 0$ the transients in the exact solution are no longer weak nor short lived. Instead, the exact solution oscillates about a mean value which decays slowly with time, and with an amplitude which also decays slowly with time; both decays are exponential but slow. At $\mu = 0$, the exact solution oscillates with constant amplitude about a mean value which does not decay with time. In the region of small μ -values $f_1(\tau)$ takes on what appears visually to be a time average of the exact solution and $f_2(\tau)$ essentially duplicates $f_1(\tau)$ except for a very short transient. The adiabatic approximation, $f_0(\tau)$, provides a reasonable picture of the average behavior of the exact solution for large values of μ, but, for small μ both $f_1(\tau)$ and $f_2(\tau)$ are definitely preferable representations of this average behavior. Only $f_3(\tau)$ fails as an approximation to the exact solution for some values of μ and large times. When $\mu \lesssim 0.6$, $f_3(\tau)$ contains a positive exponential mode which causes this solution to grow
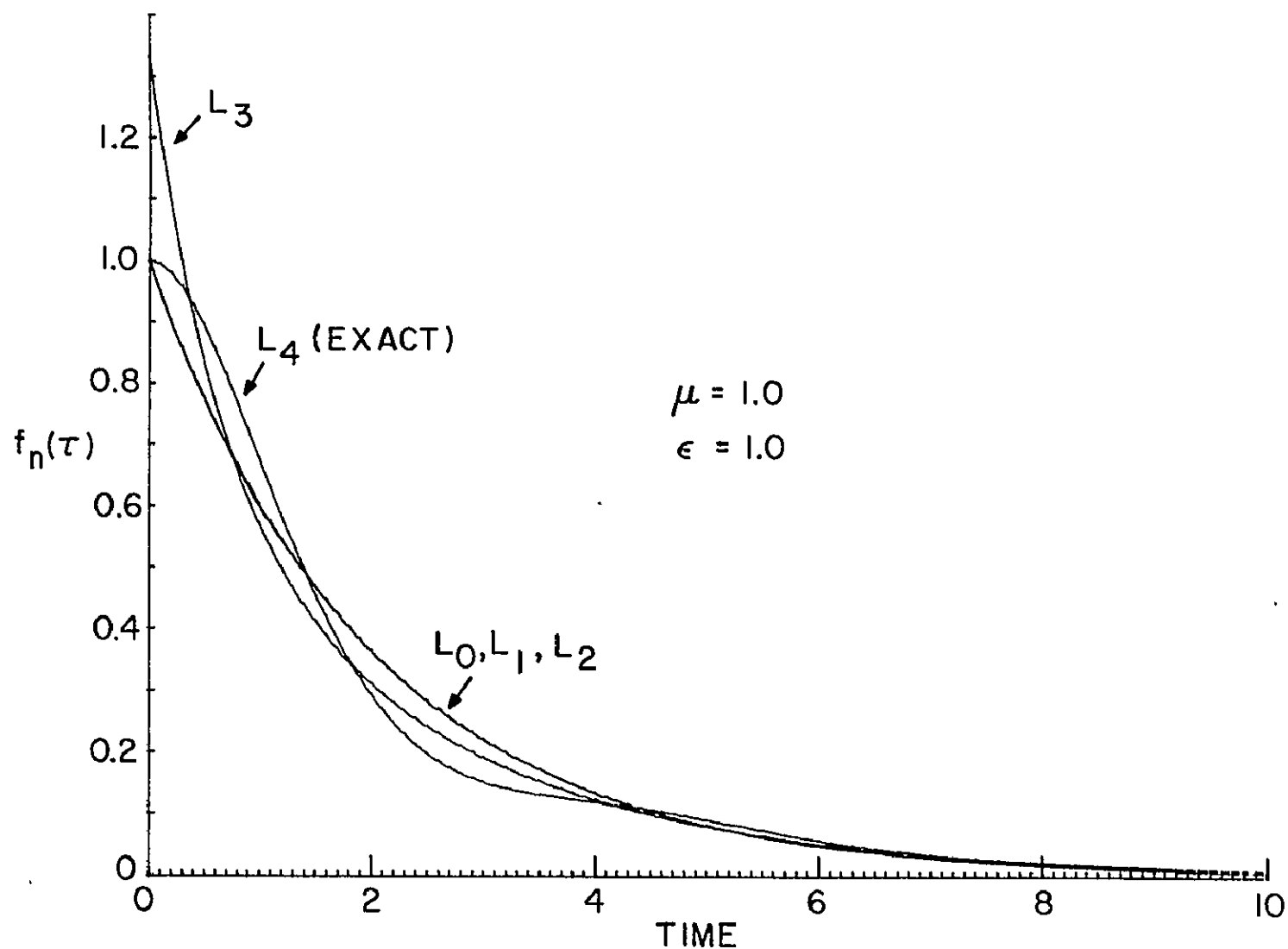
25

Figure 2. The zero'th (adiabatic) through fourth (exact) local approximations for the exactly soluble problem with a C-number kernel. This is a strong coupling (ϵ = 1) example with μ = 1.0. The zero'th, first and second local approximations are identical. Notice the weak, short-lived transients in the exact solution, following which all of the local approximations become essentially identical.
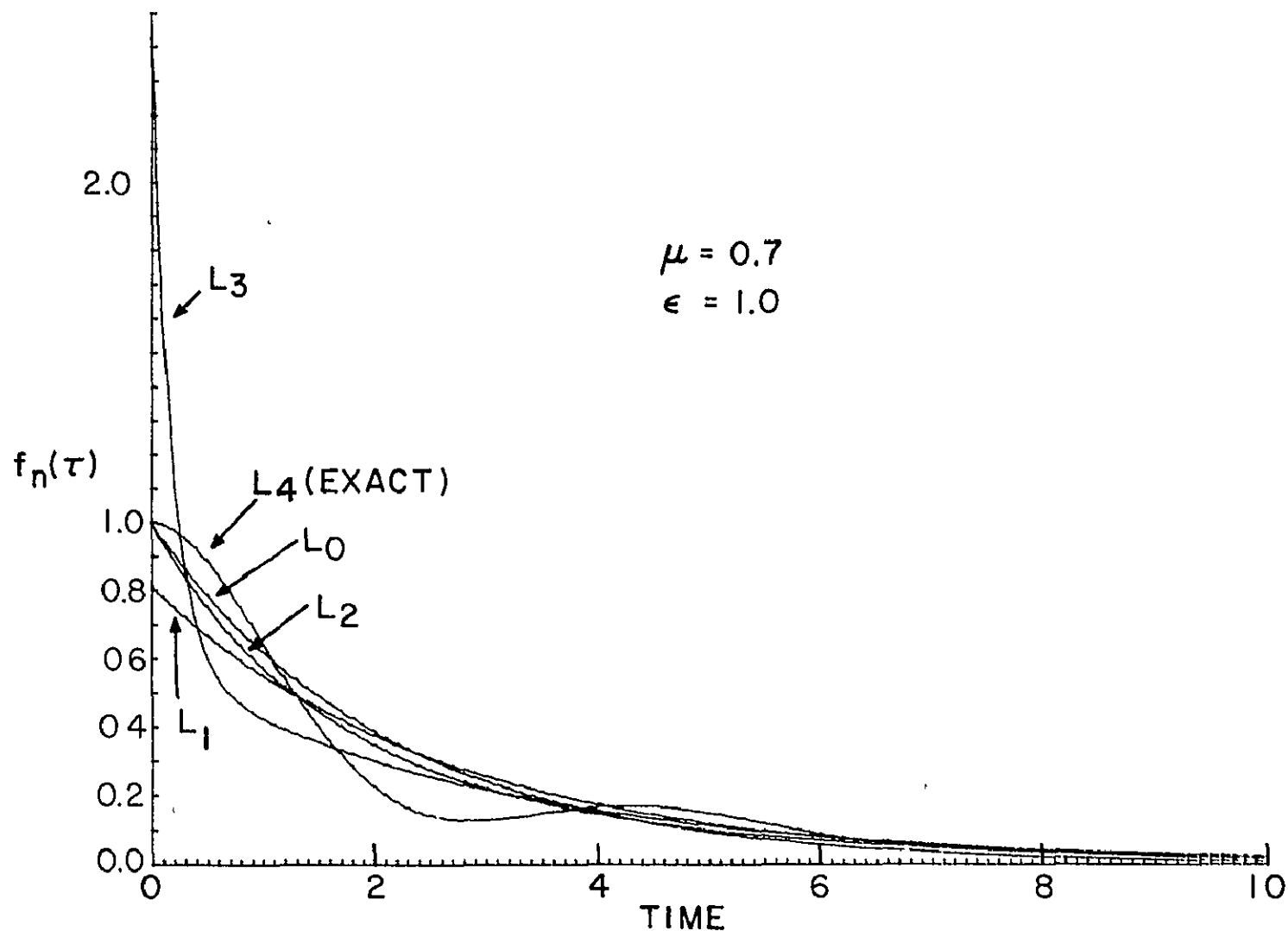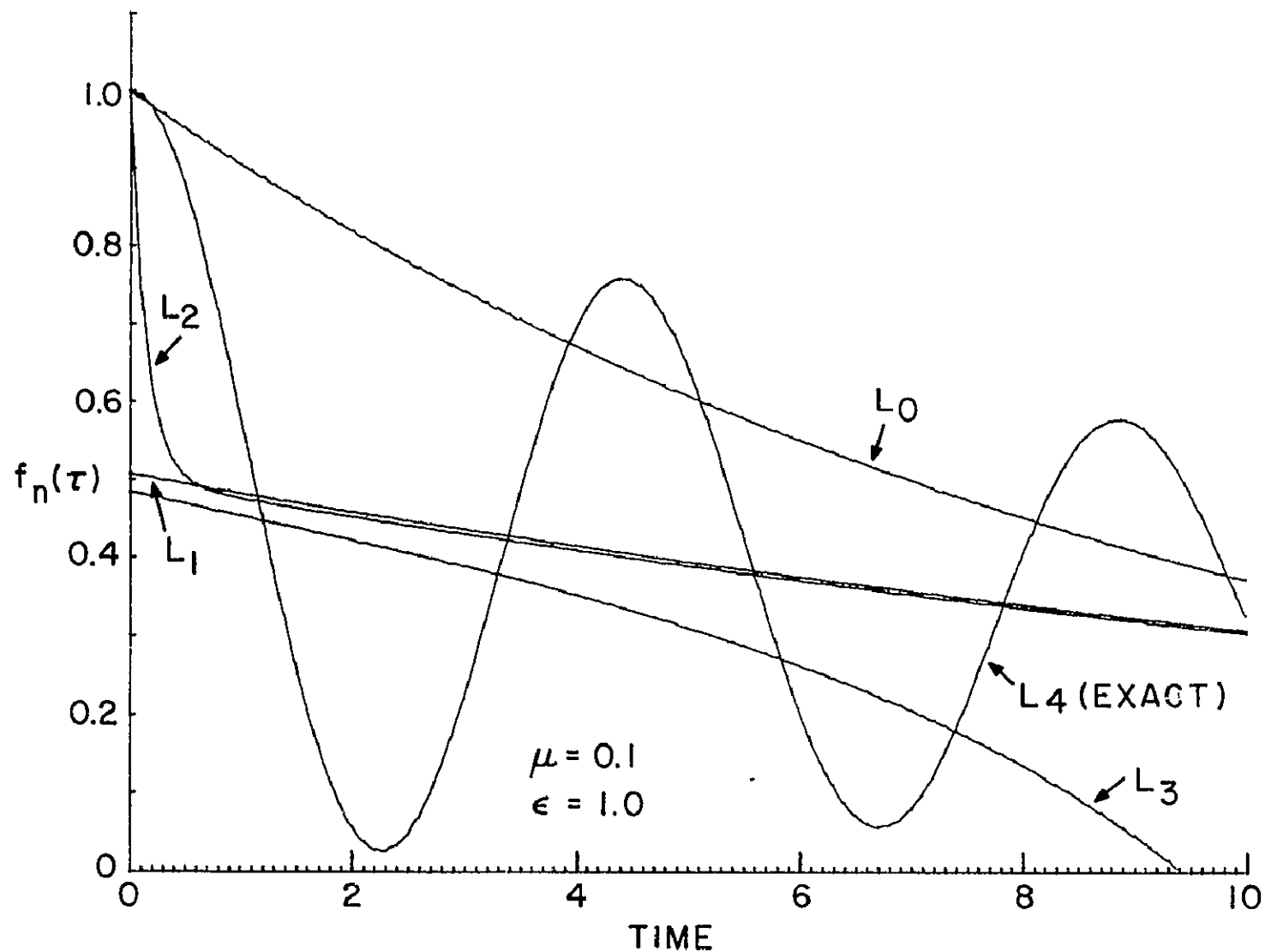
Figure 3. The zero'th (adiabatic) through fourth (exact) local approximations for the exactly soluble problem with a C-number kernel. This is a strong coupling ($\epsilon = 1$) example with $\mu = 0.7$. The individual reinitializations are readily apparent in this figure. Notice the transients in the exact solution are somewhat stronger than in the previous example. During the transients the local approximations give a reasonable picture of the average behavior of the exact solution.
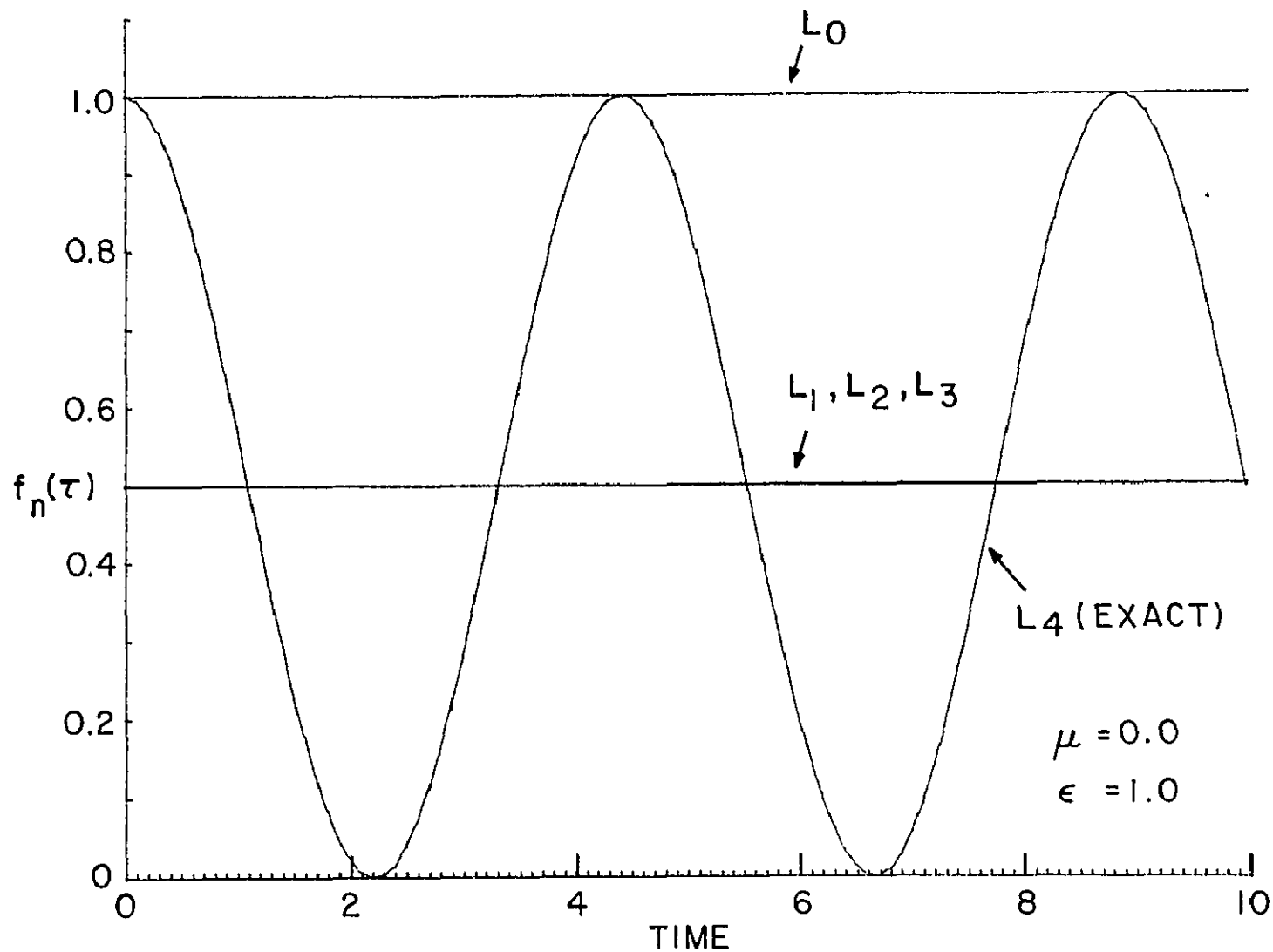
Figure 4. The zero'th (adiabatic) through fourth (exact) local approximations for the exactly soluble problem with a C-number kernel. This is a strong coupling (ϵ = 1) example with μ = 0.1. The exact solution is dominated by slowly decaying oscillations. Both the first and second local approximations give the average evolution of the exact solution; the zero'th local approximation does not appear as useful. The third local approximation contains a growing exponential mode which leads to its divergence with increasing time.

Figure 5. The zero'th (adiabatic) through fourth (exact) local approximations for the exactly soluble problem with a C-number kernel. This is a strong coupling ($\epsilon = 1$) example with $\mu = 0.0$. The exact solution oscillates trigonometrically for all time. The first, second, and third local approximations are identical and are exactly the time average of the exact solution. The zero'th local approximation gives an indication of the average behavior of the exact solution, however, its amplitude is shifted upward from the correct value.

unbounded with increasing time. This exponential growth rate decreases with decreasing μ, however, so that as μ decreases $f_3(\tau)$ remains a viable approximation for longer times. At $\mu = 0$ this growth rate goes to zero and $\dot{f}_3(\tau)$ becomes identical to $f_1(\tau)$ and $f_2(\tau)$. All three of these approximations then become equal to the constant time average of the oscillating exact solution.

We feel that these results, and those to follow, do indicate the existence of a method for generating kinetic approximations which give excellent fits to the exact solution whenever it evolves slowly, or else, give an indication of the time average behavior of the exact solution it involves rapidly. This method depends on the existence of a formal solution in Laplace space which can be simplified considerably through a Maclaurin series expansion. This Maclaurin series expansion, with its typically finite radius of convergence, is replaced by Pade' approximants in order to continue the approximation to the exact solution beyond the radius of convergence to $p = \infty$. The accuracy of the approximation for large p does not seem to be too signficant since this domain in p seems to affect only the initial short-lived transients in the local approximations. Various Pade' approximants give various representations of the solution for short times, but they are all equivalent for large times when they produce meaningful results at all. In applications of the Pade' approximants in the literature, where the exact solution is not available, the validity of the results is typically judged by computing the predictions of several approximants and comparing them. On this basis, in our application here, we would certainly reject the third local

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approximation since it diverges radically from the zero'th, first, and second approximations which all agree with each other reasonably well.

There is one further aspect of this approximation technique which is particularly interesting. In our construction of the various local approximations we have relied solely on the smallness of the Laplace variable; the parameter, ϵ, was not assumed small. To emphasize this point we have plotted the various local approximations in figures 2 through 5 for $\epsilon = 1.0$. In paper I. we have presented two examples of the more traditional reasoning which leads to the adiabatic approximation as a leading approximation in an asymptotic expansion in the small parameter, ϵ. Here we see that the adiabatic approximation, and particularly the higher order local approximations, still yield information about the average behavior of the exact solution even when ϵ is not small. Equation II.1 can be considered a model of the kinetic theory problem for a variety of physical systems.[7,8] When ϵ is small we are in the "weak coupling" regime for all of these kinetic theories. It appears that with the use of the local approximation method we should be able to learn something about the average behavior of these other physical systems in the strong coupling regime in which the particle-particle or particle-wave interaction strength and/or the particle density is not assumed asymptotically small.

IV. APPLICATION TO THE QUASI-LINEAR DIABATIC EQUATION FOR

TEST PARTICLE PROPAGATION IN MAGNETOSTATIC TURBULENCE

In this section we compute the first local approximation to the probability

distribution function which is the solution of the quasi-linear diabatic equation

constructed in paper I. for test particle propagation in magnetostatic turbulence.

As a matter of convenience, we treat equation I.24 from paper I. which we

reproduce here:

$$\frac{\partial N(\mu, \tau)}{\partial \tau} = \eta^2 (1 - \mu^2) \int_0^\tau d\lambda \, \mathcal{K}(\mu, \lambda) \, \frac{\partial^2 N(\mu, \tau - \lambda)}{\partial \mu^2} \qquad \text{II.49}$$

in which,

$$N(\mu, \tau) = \int_{-1}^{\mu} d\mu' \, \overline{f}(\mu', \tau) \qquad \text{II.50}$$

Thus, in this section we obtain numerical solutions of equation II.18 with ϵ

replaced by η^2, and with $K(\lambda)$ given by,

$$K(\lambda) = (1 - \mu^2) \, \mathcal{K}(\mu, \lambda) \, \frac{\partial^2}{\partial \mu^2} \qquad \text{II.51}$$

where,

$$\mathcal{K}(\mu, \lambda) = R(\mu\lambda) \cos \lambda \qquad \text{II.52}$$

and where $R(\zeta)$ is the two point correlation function associated with the assumed homogeneous magnetostatic turbulence. As in paper I. we adopt,

$$R(\zeta) = e^{-|\zeta|} \qquad \text{II.53}$$

Then,

$$\tilde{K}(p) = (1 - \mu^2)\, \tilde{\mathcal{K}}(\mu,\, p)\, \frac{\partial^2}{\partial \mu^2} \qquad \text{II.54}$$

in which,

$$\tilde{\mathcal{K}}(\mu,\, p) = \frac{p + |\mu|}{1 + (p + |\mu|)^2} \qquad \text{II.55}$$

The probability distribution function is obtained by numerically differentiating $N(\mu,\, \tau)$ with respect to μ.

We have chosen to treat equation I.26 for $N(\mu,\, \tau)$, as opposed to equation I.22 for $\bar{f}(\mu,\, \tau)$, since the Q-number kernel in this case can be separated into a product of a function of the dummy time variable, λ, times an operator which does not operate on the function. In this special case, the entire process of constructing the various local approximations can proceed leaving the second order differential operator untouched. The construction of the equations which generate the local approximations in this case is no more difficult than the construction presented in the previous section for the C-number kernel although the extraction of the solutions of these equations is considerably more difficult.

Since the Laplace transform of the kernel which we are considering here is the operator ratio of a first to a second order polynomial in p, the local approximation method truncates at n = 4 yielding, in that order, a partial differential equation for $N_4(\mu, \tau)$ which is third order in time, and which generates the exact solution. From equations II.37 through II.43 we find,

$$\frac{\partial}{\partial \tau} \left[\frac{\partial^2 N_4}{\partial \tau^2} + 2 |\mu| \frac{\partial N_4}{\partial \tau} + (1 + \mu^2) N_4 - \eta^2 (1 - \mu^2) \frac{\partial^2 N_4}{\partial \mu^2} \right]$$

II.56

$$= \eta^2 |\mu| (1 - \mu^2) \frac{\partial^2 N_4}{\partial \mu^2}$$

subject to,

$$\frac{\partial N_4(\mu, 0)}{\partial \tau} = 0 \qquad \frac{\partial^2 N_4(\mu, 0)}{\partial \tau^2} = \eta^2 (1 - \mu^2) \frac{\partial^2 N_4(\mu, 0)}{\partial \mu^2} \qquad \text{II.57}$$

On comparing, we find that we have regained equations I.50 and I.51 from paper I. which were obtained by repeated differentiation with respect to time of equation II.49. We have also shown in paper I. that equations I.50 and I.51 are equivalent to equations I.44 through I.49 which were used there to obtain numerical evaluations of the exact quasi-linear diabatic solutions for several interesting initial conditions. Thus, the numerical solutions presented in paper I. can be viewed as the fourth local approximation to the probability distribution function. We further constructed the adiabatic approximation in paper I. which we found is governed by,

$$\frac{\partial N_0}{\partial \tau} = \eta^2 \, |\mu| \, \left(\frac{1 - \mu^2}{1 + \mu^2}\right) \, \frac{\partial^2 N_0}{\partial \mu^2} \qquad\qquad \text{II.58}$$

and, in addition we presented one numerical evaluation of the solution of this equation for a narrow parallel beam injection. In the context of the present discussion, equation II.58 and the example of a solution given in paper I. represent the zero'th local approximation to the test particle problem. We have discussed the failure of the zero'th local approximation in paper I.. In this section we will present the results of a numerical evaluation of the first local approximation to the probability distribution function for the same initial conditions considered in paper I. in order to demonstrate the improved level of approximation in going from n = 0 to n = 1.

a). The Laplace modes.

We will begin our discussion of the first local approximation applied to the test particle propagation problem through a discussion of the first local approximation to the Laplace modes which we introduced in paper I.. In general we will show that the first local approximation to the Laplace modes and the related eigenvalues is a significant improvement to the zero'th local approximation for several of the low order Laplace modes when the Laplace variable is near zero.

We have seen in paper I. (equation I.64), that the Laplace modes are generated by the eigenvalue equation,

$$\frac{\partial}{\partial \mu} (1 - \mu^2) \tilde{K}(\mu, p) \frac{\partial \psi_m(\mu, p)}{\partial \mu} + \lambda_m(p) \psi_m(\mu, p) = 0 \qquad \text{II.59}$$

We showed generally that $\tilde{K}(\mu, p)$ is a positive definite function of μ when p is real and positive (this property is explicitly demonstrated in equation II.53 for the special case being considered here). Thus, the Sturm-Liouville method could be applied to generate a complete set of non-degenerate orthogonal eigenfunctions with a discrete set of eigenvalues. Several of these eigenfunctions and eigenvalues were computed numerically and presented in paper I. for various values of p.

The n'th local approximation to the eigenfunctions and eigenvalues generated by equation II.59 is obtained by substituting the n'th convergent, $\tilde{K}_n(\mu, p)$, in that equation for $\tilde{K}(\mu, p)$. The zero'th convergent to $\tilde{K}(\mu, p)$ is

$$\tilde{K}_0(\mu, p) = \frac{|\mu|}{1 + \mu^2} \qquad \text{II.60}$$

This expression explicitly demonstrates a general feature of the zero'th convergent to $\tilde{K}(\mu, p)$ which was pointed out in paper I.; i.e., the zero'th convergent contains a zero at $\mu = 0$ which leads to doubly degenerate, discontinuous eigenfunctions in the zero'th local, or adiabatic approximation level. The failure of this level of approximation was shown in paper I. to be directly attributable to the discontinuous nature of these eigenfunctions.

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In Appendix B of this paper we attempt a systematic expansion in powers of p of the eigenfunctions and eigenvalues using the adiabatic quantities as the leading approximation and also using degenerate perturbation analysis to remove the degeneracy in this leading approximation. There we find unbounded first order corrections to the adiabatic eigenvalues due to the discontinuities in the adiabatic eigenfunctions. Thus, the discontinuities in the adiabatic eigenfunctions lead to the failure of another standard perturbation method.

The first convergent to $\tilde{\mathcal{K}}(\mu, p)$ is,

$$\tilde{\mathcal{K}}_1(\mu, p) = \frac{|\mu|}{1 + \mu^2} + \left[\frac{1 - \mu^2}{(1 + \mu^2)^2}\right] p \qquad \text{II.61}$$

This convergent is positive definite in μ for p real and positive and therefore does not lead to the problems contained in the zero'th local approximation which were discussed above. On substituting this convergent into equation II.59 for $\tilde{\mathcal{K}}(\mu, p)$, we find that the low order eigenfunctions and eigenvalues are very well approximated for small p. In figures 6 and 7 we plot the differences between the exact and first local ψ_1 and ψ_2 for several small values of p, and in Table I the exact and first local eigenvalues are compared. As in paper I., these results were obtained through numerical integration using a Runge-Kutta integration routine and the "shot-gun method" on equation II.59 containing the first convergent instead of $\tilde{\mathcal{K}}(\mu, p)$. All of these comparisons have been made using the exponential correlation function given by equation II.53
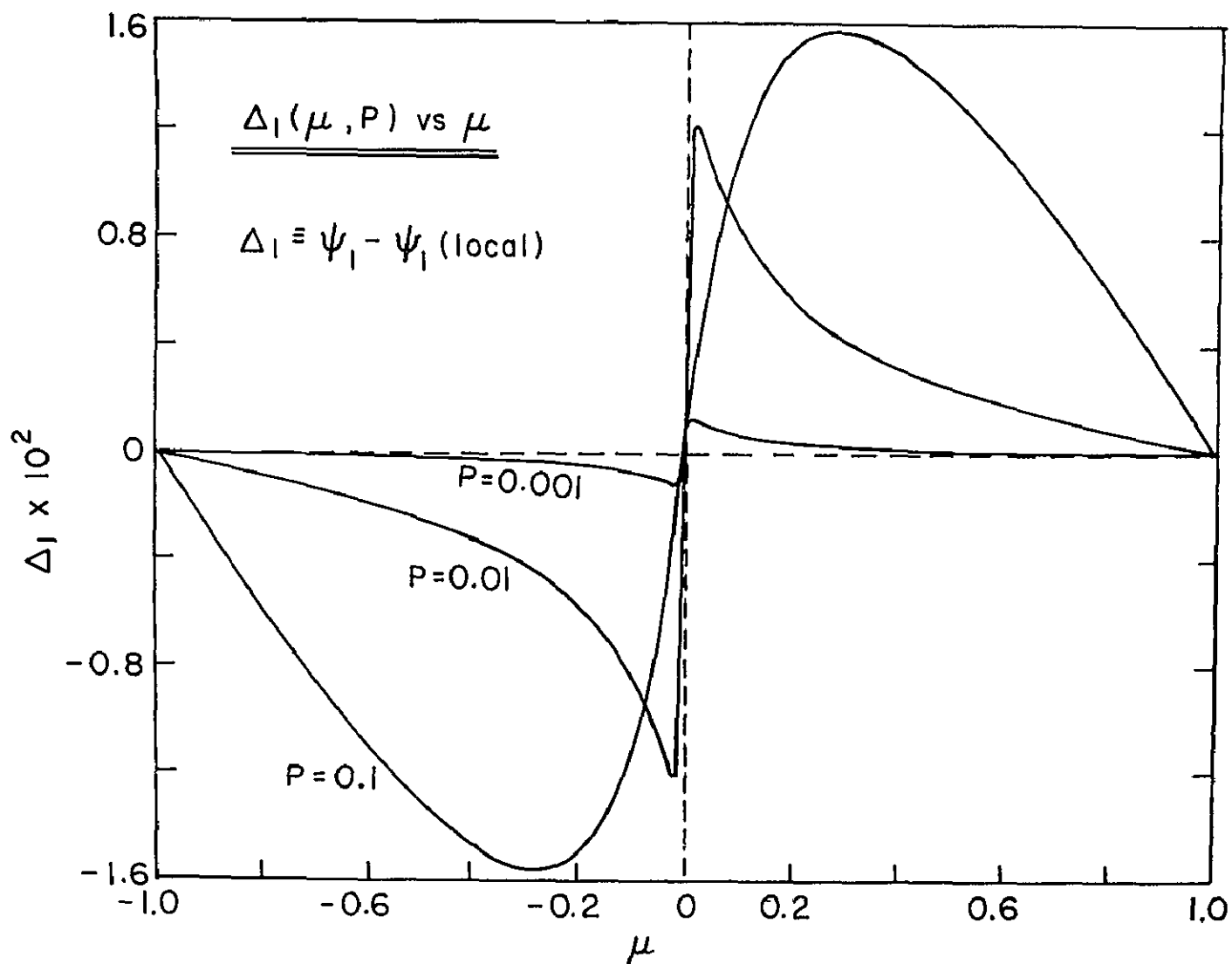
37

Figure 6. The difference between the first Laplace mode and its first local approximation is shown as a function of μ. Notice that the difference has been multiplied by a factor of 100 in this figure, and that the difference generally decreases with decreasing Laplace variable, p.
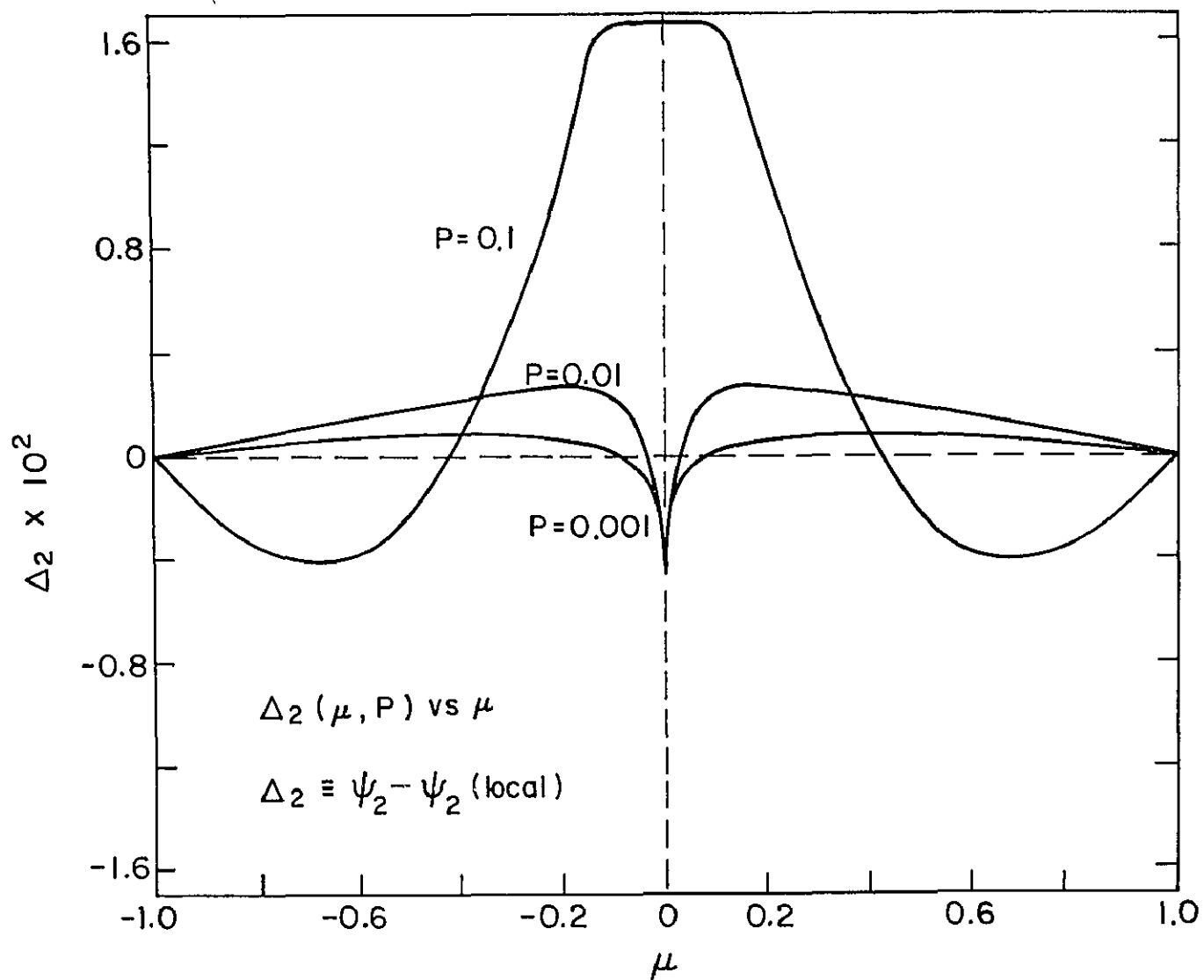
Figure 7. The difference between the second Laplace mode and its first local approximation is shown as a function of μ. Notice that the difference has been multiplied by a factor of 100 in this figure, and that the difference generally decreases with decreasing Laplace variable, p.

Table I

P	λ_1	λ_1 (local)	λ_2	λ_2 (local)
0.1	0.566	0.576	2.640	2.675
0.01	0.2750	0.2753	2.360	2.360
0.001	0.1705	0.1705	2.317	2.3175

b). Reinitialization

We obtain the equation which generates the first local approximation to

$N(\mu, \tau)$ by using the kernel given by equation II.51 in equation II.18. We find,

$$\frac{\partial}{\partial \tau} \left[N_1 + \eta^2\, \emptyset_1(\mu)\, \frac{\partial^2 N_1}{\partial \mu^2} \right]$$

II.62

$$= \eta^2\, \emptyset_0(\mu)\, \frac{\partial^2 N_1}{\partial \mu^2} - \eta^2\, \delta(\tau)\, \emptyset_1(\mu)\, \frac{\partial^2 N_1}{\partial \mu^2}$$

in which $\emptyset_n(\mu) = (1 - \mu^2)\, m_n(\mu)$ and,

$$m_0(\mu) = \frac{|\mu|}{1 + \mu^2} \qquad m_1(\mu) = -\frac{1 - \mu^2}{(1 + \mu^2)^2}$$

II.63

We have found numerical solutions to this equation by first determining the

reinitialized conditions from the initial conditions through,

$$N_1(\mu,\, 0^+) + \eta^2\, \emptyset_1(\mu)\, \frac{d^2 N_1(\mu,\, 0^+)}{d\mu^2} = N(\mu,\, 0)$$

II.64

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and then by using this result for the initial condition on $N_1(\mu, \tau)$ to integrate the coupled system of equations,

$$\frac{\partial G}{\partial \tau} = \eta^2 \mathcal{D}_0(\mu) \; \frac{\partial^2 N_2}{\partial \mu^2} \qquad\qquad \text{II.65}$$

and,

$$G = N_1 + \eta^2 \, \mathcal{D}_1(\mu) \; \frac{\partial^2 N_1}{\partial \mu^2} \qquad\qquad \text{II.66}$$

which are equivalent, with some further boundary-initial conditions discussed below, to equation II.62 for positive τ.

Some examples of reinitialization are given in figures 8, 9 and 10 for the exponential correlation function. These results have been obtained by numerical-cally integrating equation II.64 to obtain $N_1(\mu, 0^+)$ given $N(\mu, 0)$, and then differentiating that result with respect to μ in order to obtain $\overline{f}_1(\mu, 0^+)$. In each case the initial condition is considerably smoothed in the process of obtaining the reinitialized condition. It is our impression that this reinitialization procedure accomplishes at least some of the smoothing which is provided by the initial transients in the exact solutions of the quasi-linear diabatic equation.

c). The probability distribution function

The boundary-initial conditions which are necessary to integrate equation II.64 through II.66 are determined as follows: From the definition of $N(\mu, \tau)$
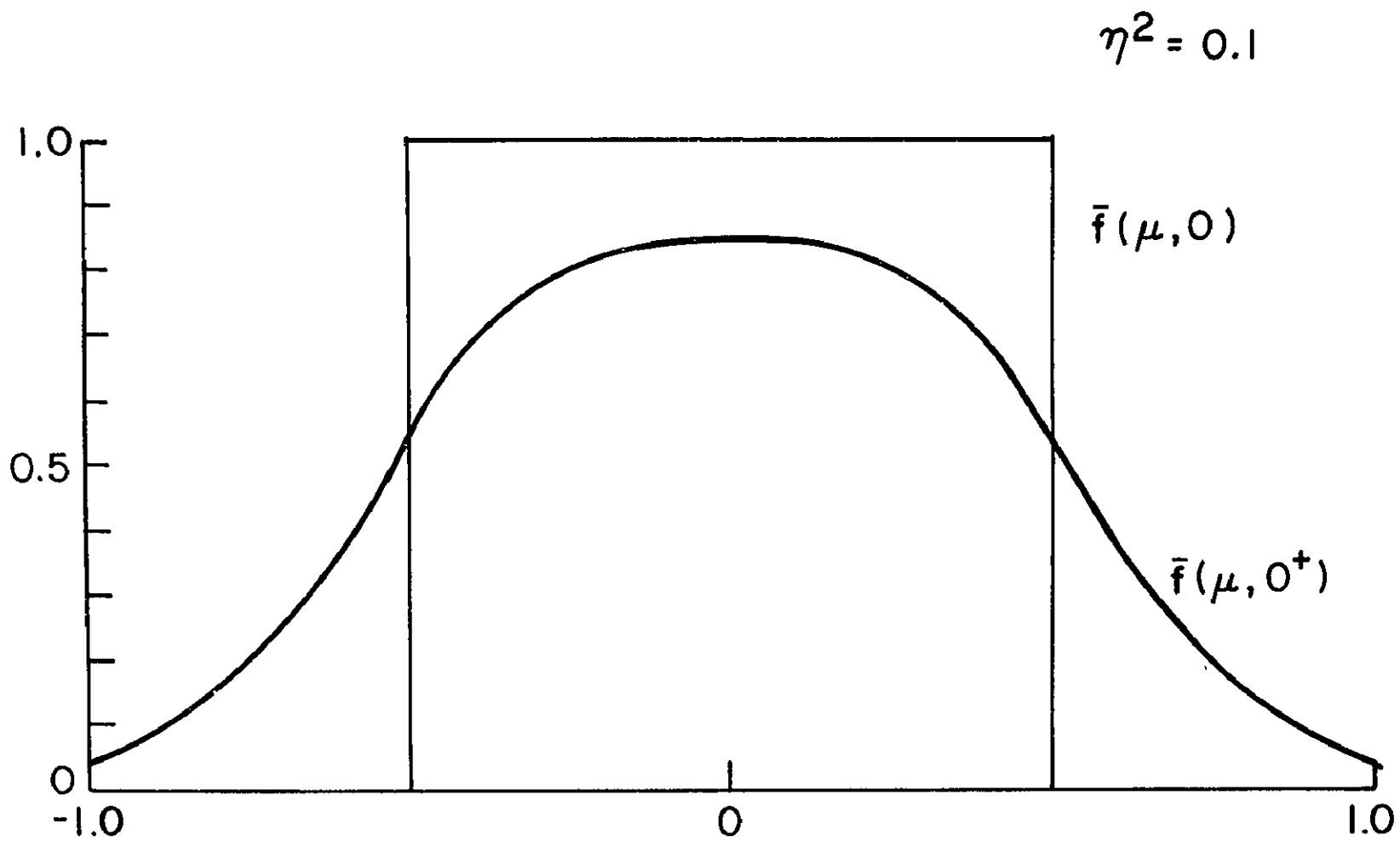
41

$$\eta^2 = 0.1$$

$$\bar{f}(\mu, 0)$$

$$\bar{f}(\mu, 0^+)$$

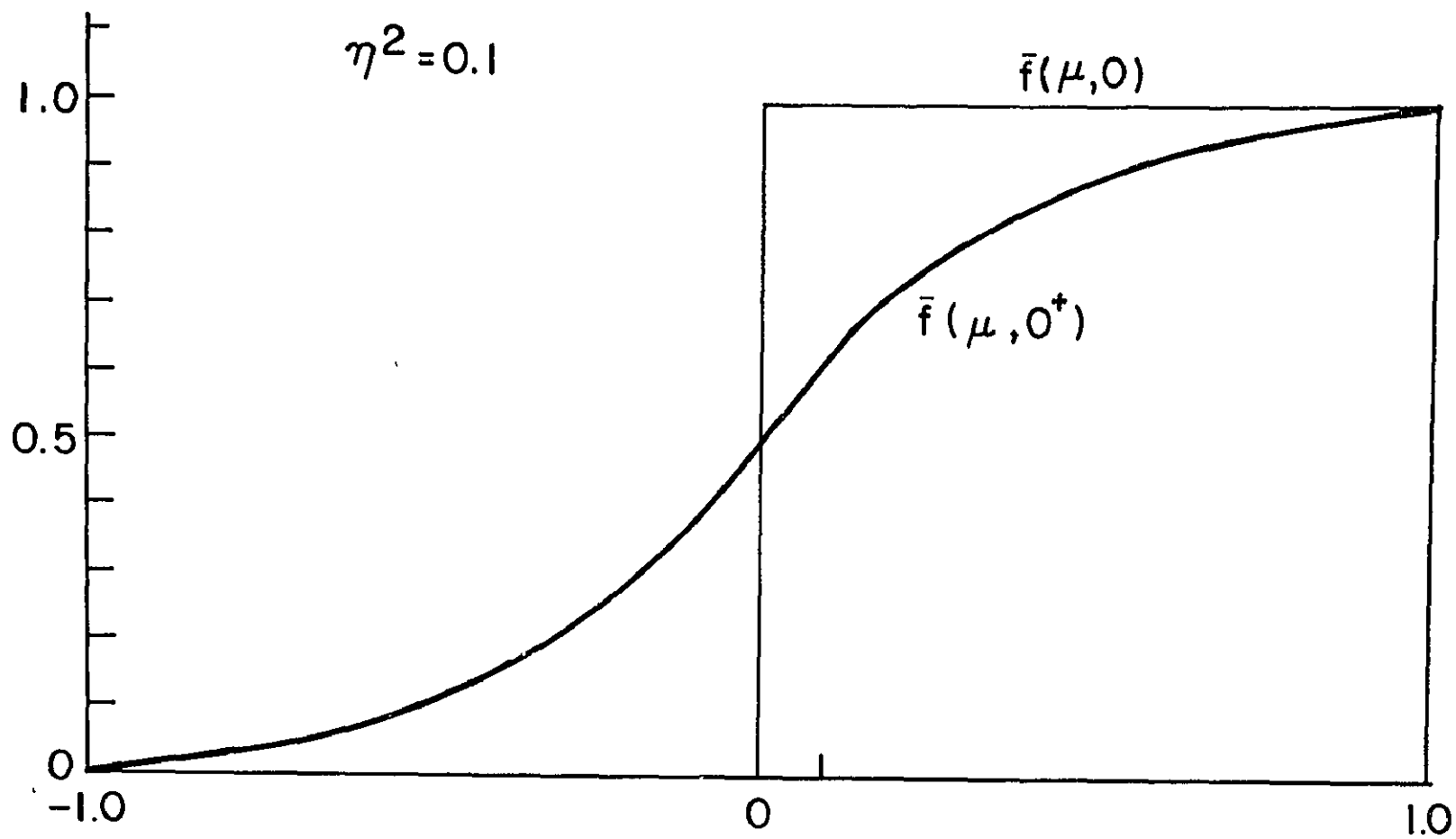Figure 8. Reinitialization of f(μ, 0) leads to f(μ, 0^+).

Figure 9. Reinitialization of $f(\mu, 0)$ leads to $f(\mu, 0^+)$.

$\eta^2 = 0.1$
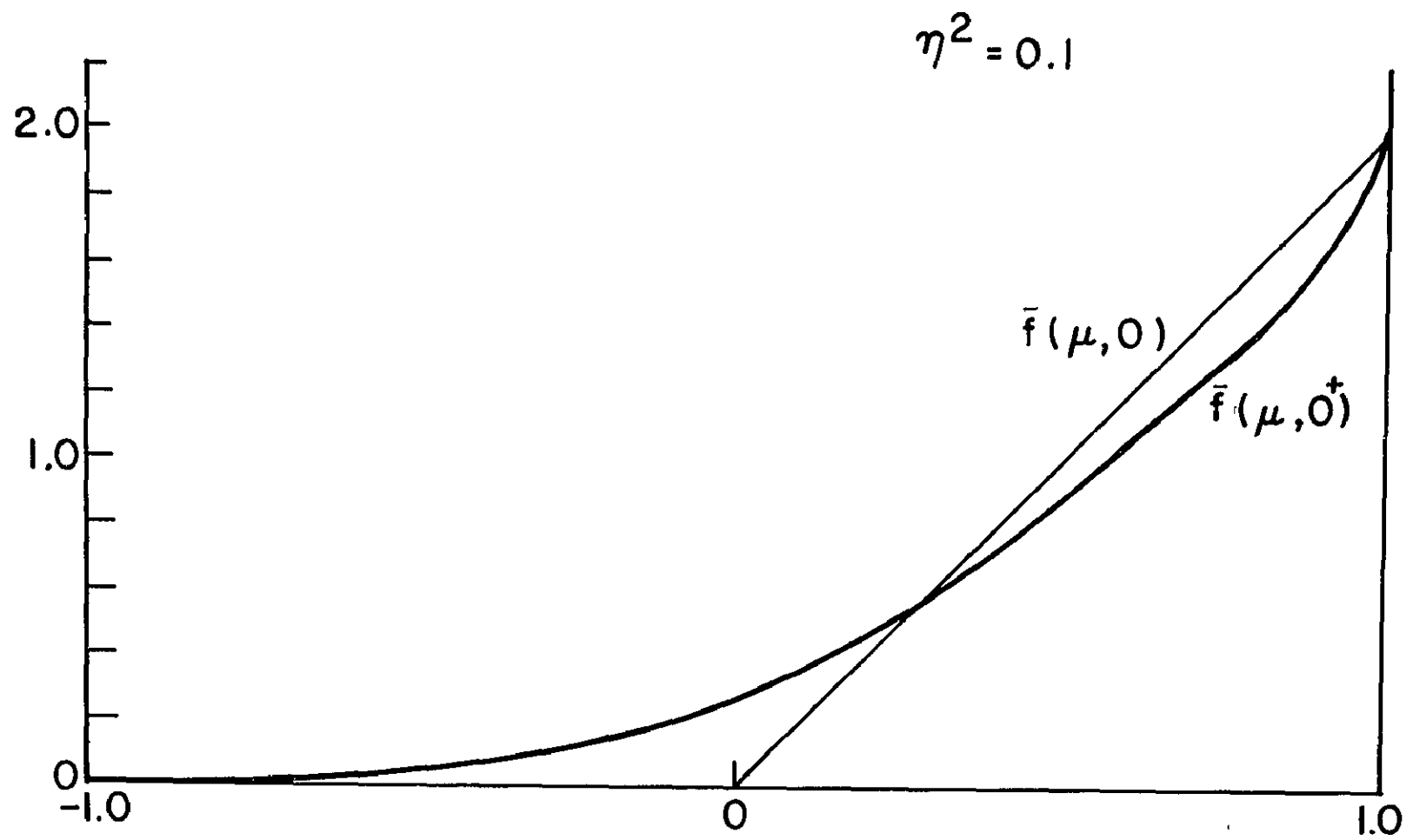
Figure 10. Reinitialization of $f(\mu, 0)$ leads to $f(\mu, 0^+)$.

given in equation II.50 we see that $N(-1, \tau) = 0$, and from equations II.64 through II.66 we further see that $N(1, \tau) = N(1, 0)$ because $\emptyset_0(1) = \emptyset_1(1) = 0$. Since equation II.62 is homogeneous in $N(\mu, \tau)$ we choose $N(1, \tau) = 1$ without loss of generality. Thus, we have established the boundary conditions on $N(\mu, \tau)$. The boundary conditions on $G(\mu, \tau)$ are given by $G(\pm 1, \tau) = N(\pm 1, \tau)$ from equation II.66 because $\emptyset_1(\pm 1) = 0$, and the initial condition on $G(\mu, \tau)$ is given by $G(\mu, 0^+) = N(\mu, 0^+)$ from equations II.64 and II.66.

We have rewritten equations II.65 and II.66 as difference equations, and have used an implicit-explicit integration routine to determine $N(\mu, \tau)$. Given $N(\mu, \tau)$, we compute $G(\mu, \tau + \Delta\tau)$ from equation II.11, and then through an implicit matrix inversion, we determine $N(\mu, \tau + \Delta\tau)$ from equation II.14. This procedure is repeated in order to march forward in time. We have used the exponential correlation function in order to compute $\emptyset_0(\mu)$ and $\emptyset_1(\mu)$.

In figure 11, we present the first of these numerical results for a parallel beam injection. The initial condition in this figure is a guassian in μ, centered at $\mu = 1$; it is identical to the initial condition which was used to obtain the quasi-linear diabatic solution presented in figure 3 of paper I. These two figures should be compared carefully in order to appreciate the accuracy of the local approximation in this case. To facilitate this comparison, we also present superpositions of both solutions at one and two Larmor periods in time in figures 12 and 13 respectively. In each of these drawings, the continuous line
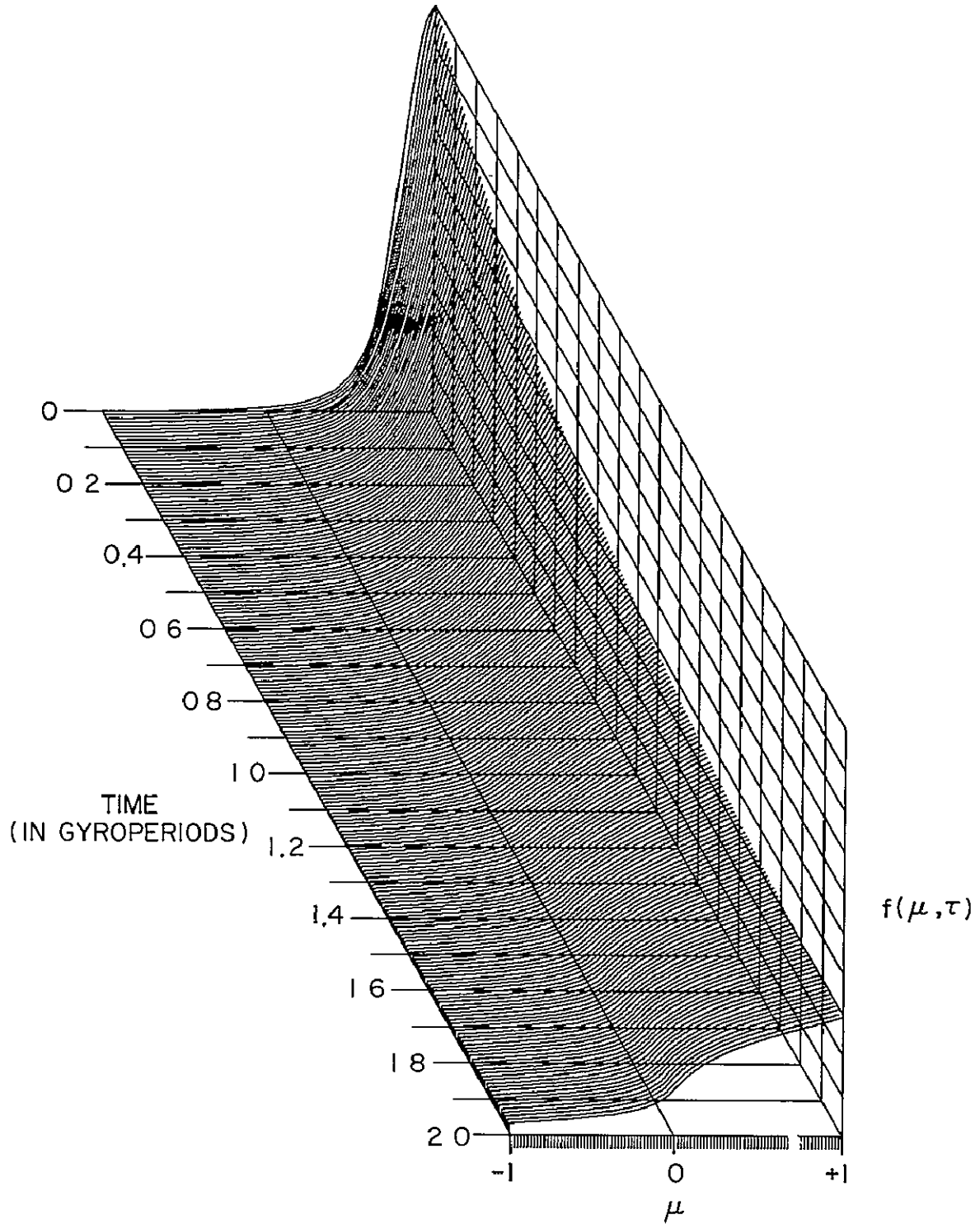
Figure 11. The first local approximation to the quasi-linear gyro-phase averaged probability distribution function for the evolution of a narrow beam injected parallel to the mean magnetic field. Compare with figure 3 of paper I.

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$\tau = 1.0$ GYRO PERIODS
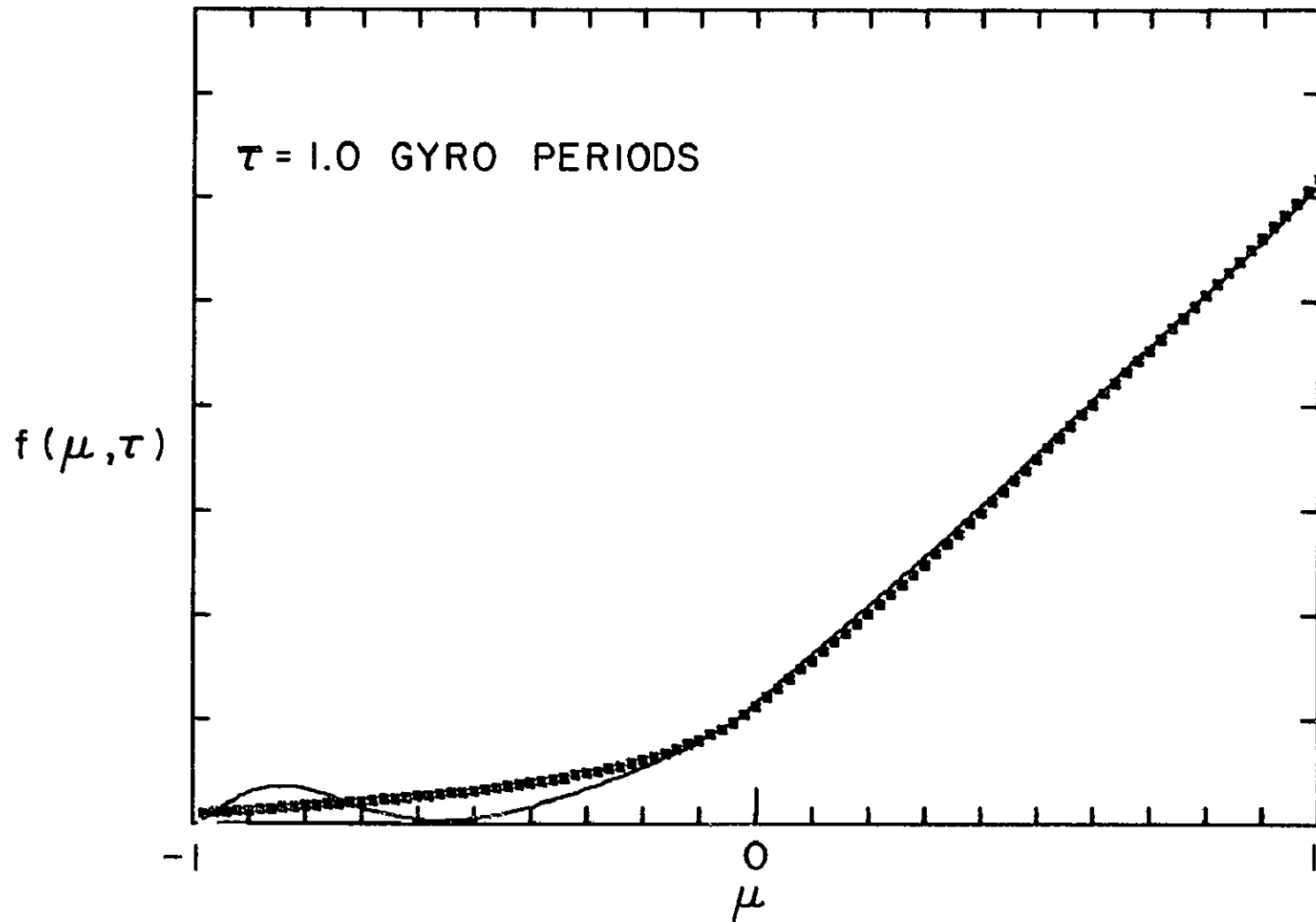
$f(\mu,\tau)$

-1

0

1

μ

Figure 12. A direct comparison, at one Larmor period, of the quasi-linear gyro-phase averaged probability distribution function with its first local approximation for the narrow parallel beam injection of figure 11. The squares give the first local approximation.
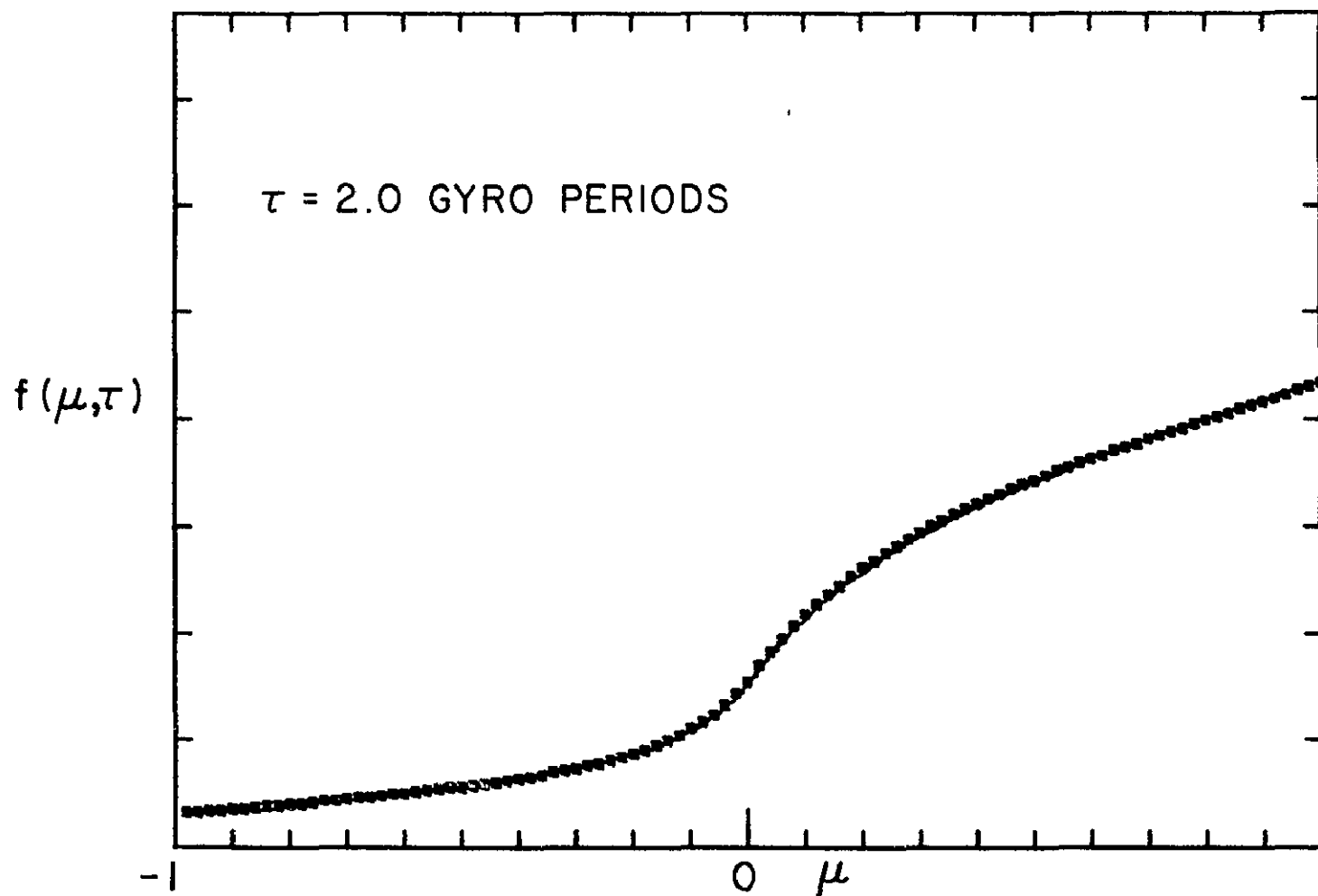
Figure 13. A direct comparison, at two Larmor periods, of the quasi-linear gyro-phase averaged probability distribution function with its first local approximation for the narrow parallel beam injection of figure 11. The squares give the first local approximation.

curve represents the quasi-linear diabatic solution, and the curve made up of small squares is its local approximation. Figure 13 does contain both solutions but the overlap of the solutions is so good, that the continuous line is not visible. Figure 13 should also be compared with figure 6 of paper I.. This comparison provides an example of the visual evidence which has lead us to an examination of the normal mode expansions of these solutions, and then, has lead us further to the local approximation method. It seems clear from this comparison that the long time structure of the probability distribution function is dominated by ψ_1 (μ, p) with small values of p. In this case, the effect of reinitialization is slight, but nevertheless, can be seen in figure 11.

Figure 14 contains the local approximation to the probability distribution function for the cross field beam injection presented in figure 4 of paper I. In this case, the effect of reinitialization is clearly visible. Notice that, although the local approximation does not contain the dramatic initial transients in the quasi-linear diabatic solution, as soon as these transients decay, the two solutions become almost identical. A direct comparison of these two solutions at one, two and three Larmor periods is presented in figures 15, 16 and 17 respectively with the same annotation as that used in the previous example. Once again, the accuracy of the local approximation is remarkable, and also, through a comparison of these figures with figure 7 of paper I. further motivation for the normal mode expansion can be obtained. Notice that by three Larmor
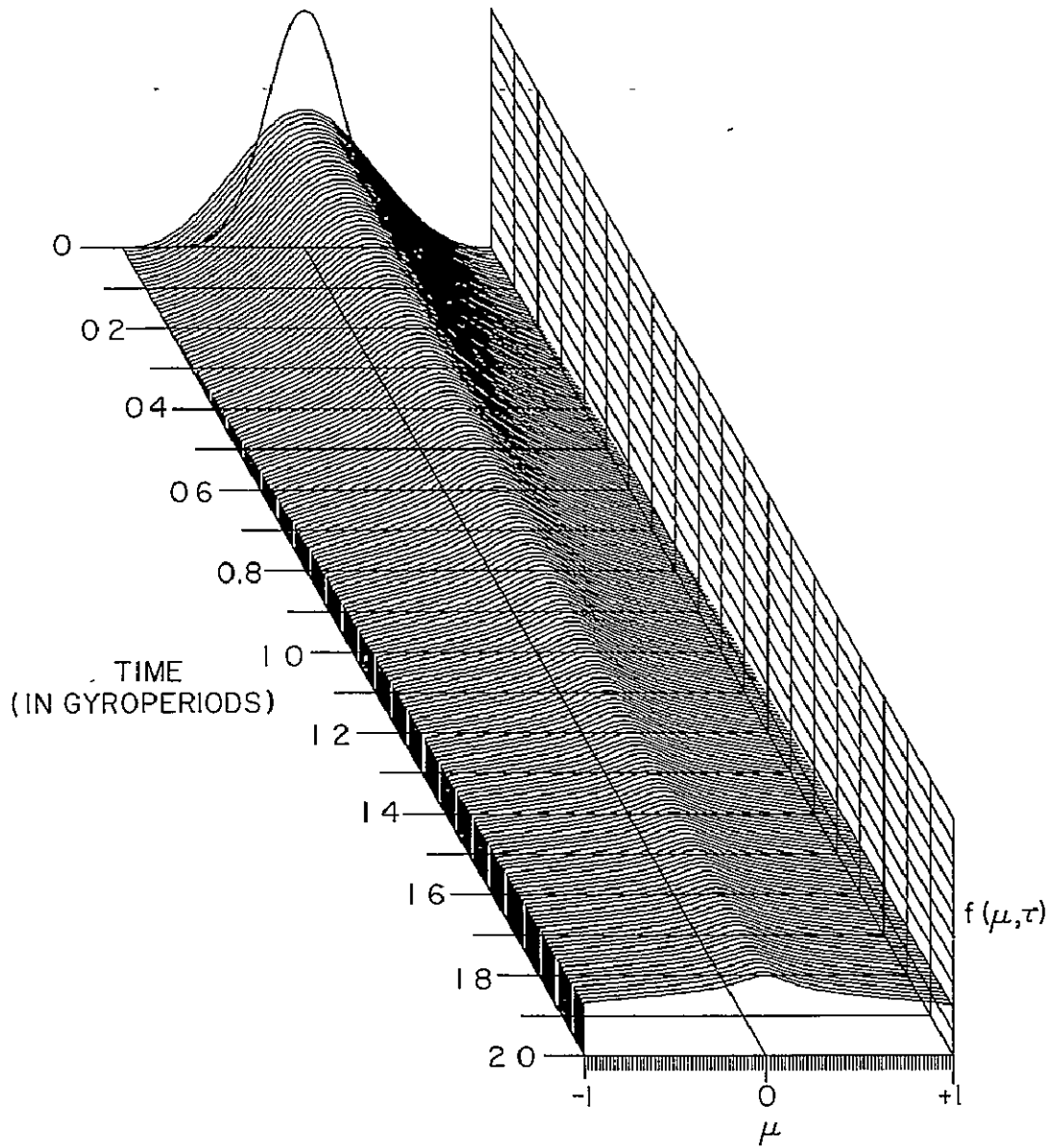
Figure 14. The first local approximation to the quasi-linear gyro-phase
averaged probability distribution function for the evolution of a narrow
beam injected across the mean magnetic field. Compare with figure 4
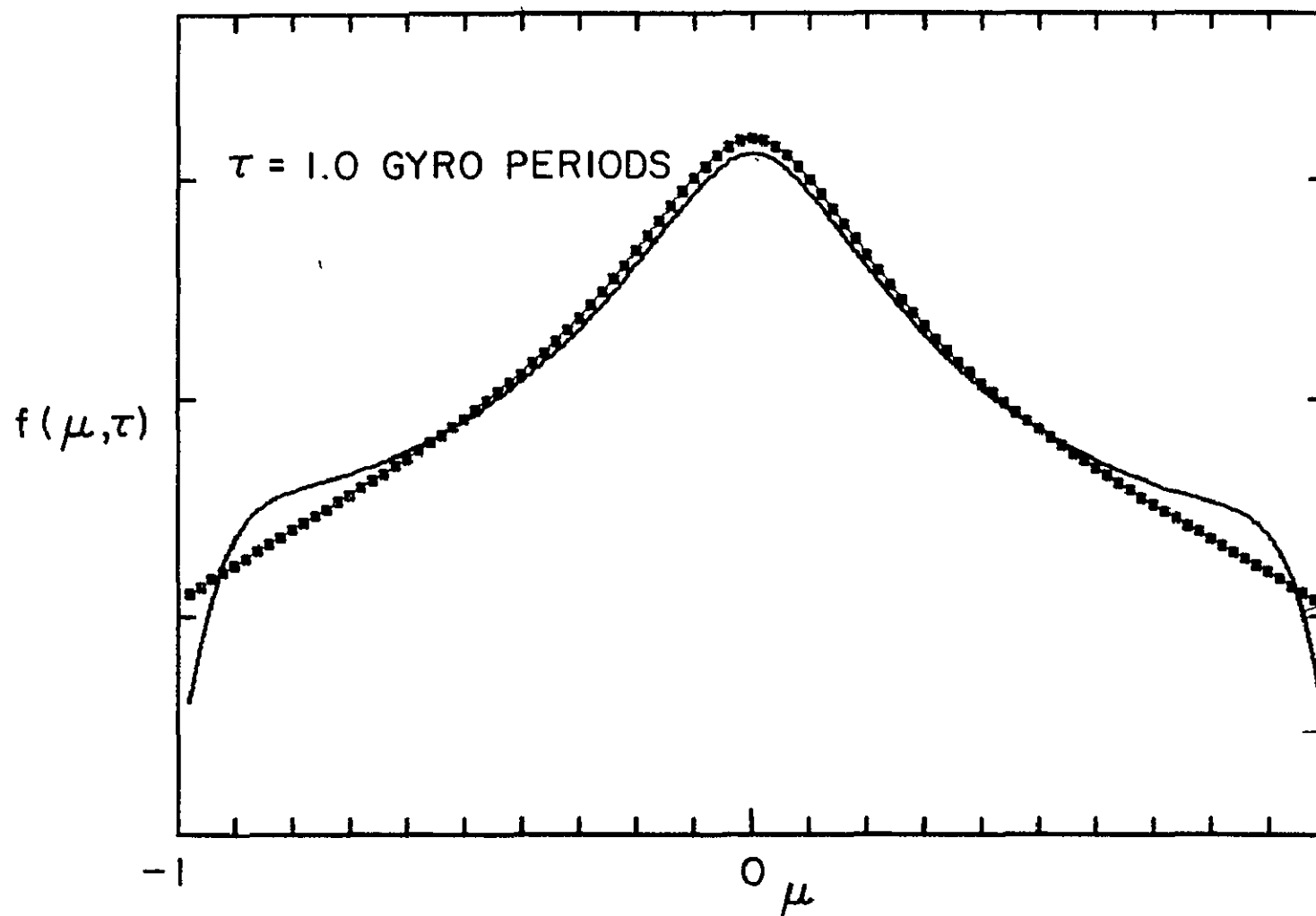of paper I.

$\tau = 1.0$ GYRO PERIODS

$f(\mu,\tau)$

-1 0 μ 1

Figure 15. A direct comparison, at one Larmor period, of the quasi-linear gyro-phase averaged probability distribution function with its first local approximation for the narrow cross field beam injection of figure 14. The squares give the first local approximation.

τ = 2.0 GYRO PERIODS

$f(\mu, \tau)$

-1 0   1

μ

Figure 16. A direct comparison, at two Larmor periods, of the quasi-linear gyro-phase averaged probability distribution function with its first local approximation for the narrow cross field beam injection of figure 14. The squares give the first local approximation.

τ = 3.0 GYRO PERIODS

$f(\mu,\tau)$

−1

0

μ

Figure 17. A direct comparison, at three Larmor periods, of the quasi-linear gyro-phase averaged probability distribution function with its first local approximation for the narrow cross field beam injection of figure 14. The squares give the first local approximation.
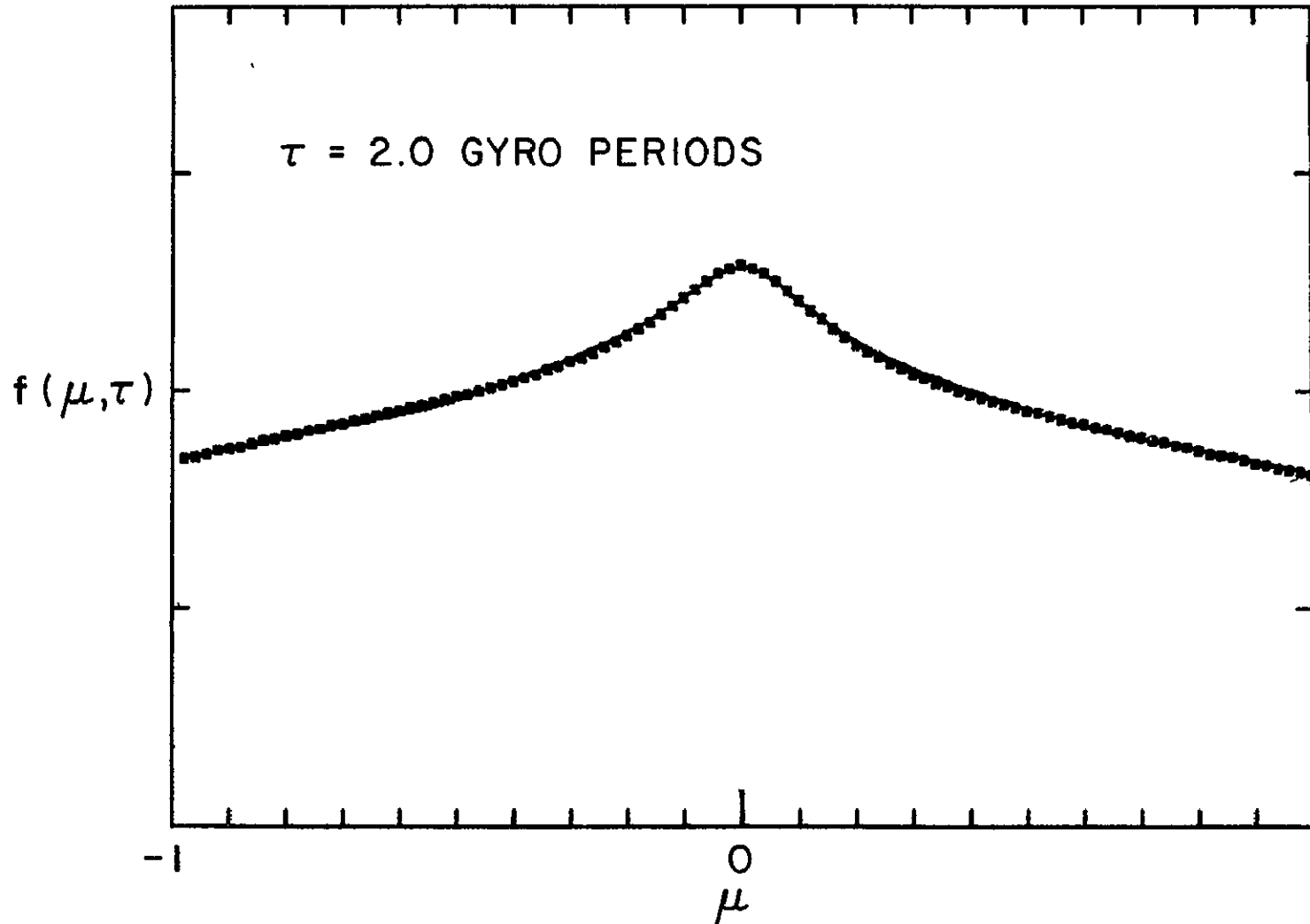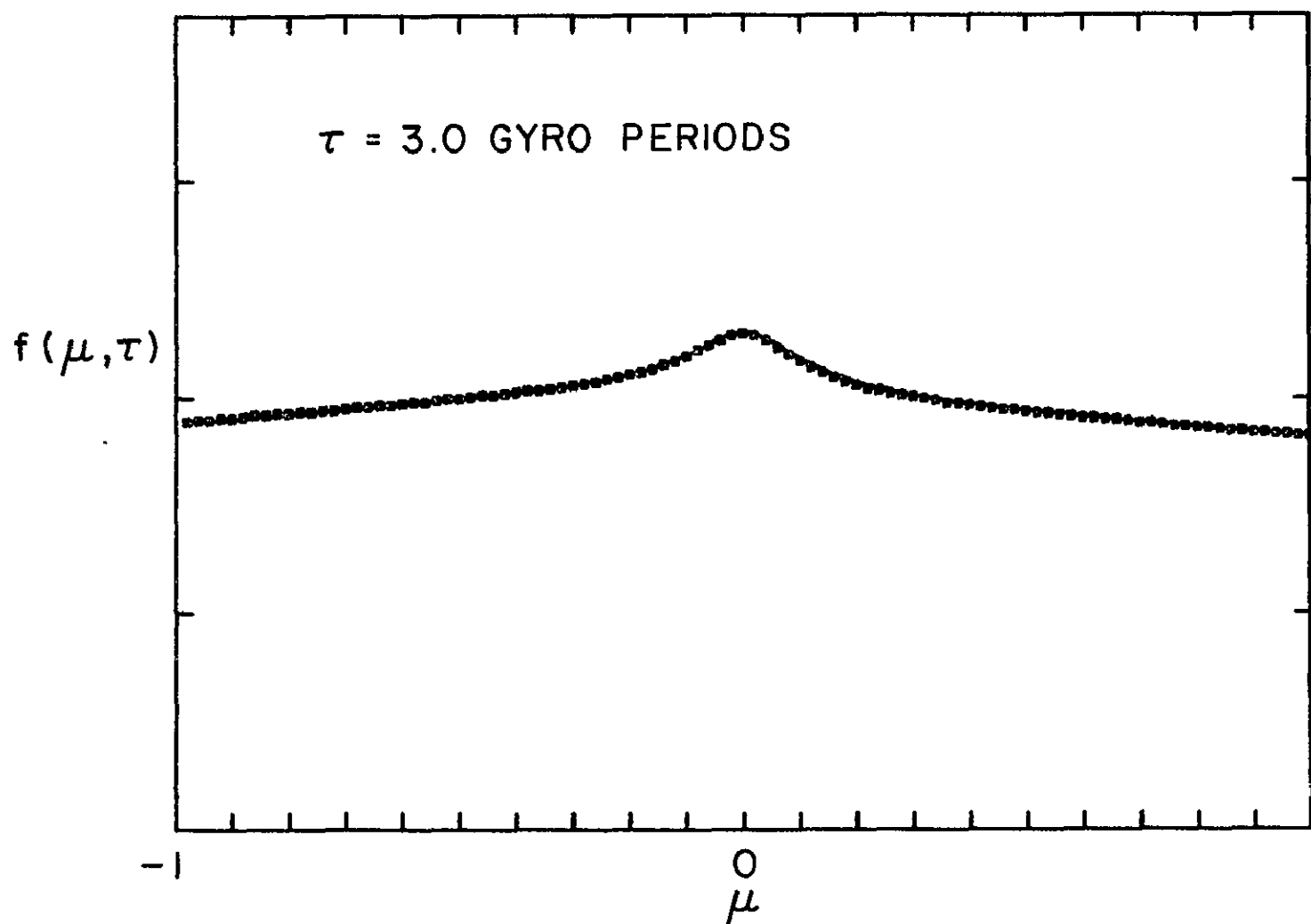
periods the probability distribution function has almost reached isotropy (μ-independence).

In figure 18 we present the local approximation to the probability distribution function for the broad parallel beam injection presented in figure 5 of paper I. Once again the effect of reinitialization and the lack of initial transients are readily apparent. Because of the development of a numerical instability in the quasi-linear diabatic solution beyond $\tau \cong 1.0$ Larmor period (see paper I.), we present only one direct comparison of these solutions at one Larmor period in figure 19. We have also carried the local approximation to this solution out to large times. In figure 20 we have plotted the local approximation every two gyro-periods from four to sixteen Larmor periods. As time increases, this solution tends toward the isotropic distribution in which the levels of the solution on either side of the steep transition region near $\mu = 0$ become equal. There is some evidence from this figure that the slope of the solution in the vicinity of $\mu = 0$ is increasing with time. Remember that in the adiabatic or diffusion approximation the slope of the probability distribution function quickly develops an infinite value at $\mu = 0$. In the next paper in this series we will show that this slope, both in the quasi-linear diabatic solution and its local approximation, approaches a large but finite value at $\mu = 0$ as the time goes to infinity. It is the development of this large slope at $\mu = 0$ which will eventually lead us to suspect the quasi-linear approximation to the probability distribution function as a viable leading approximation.
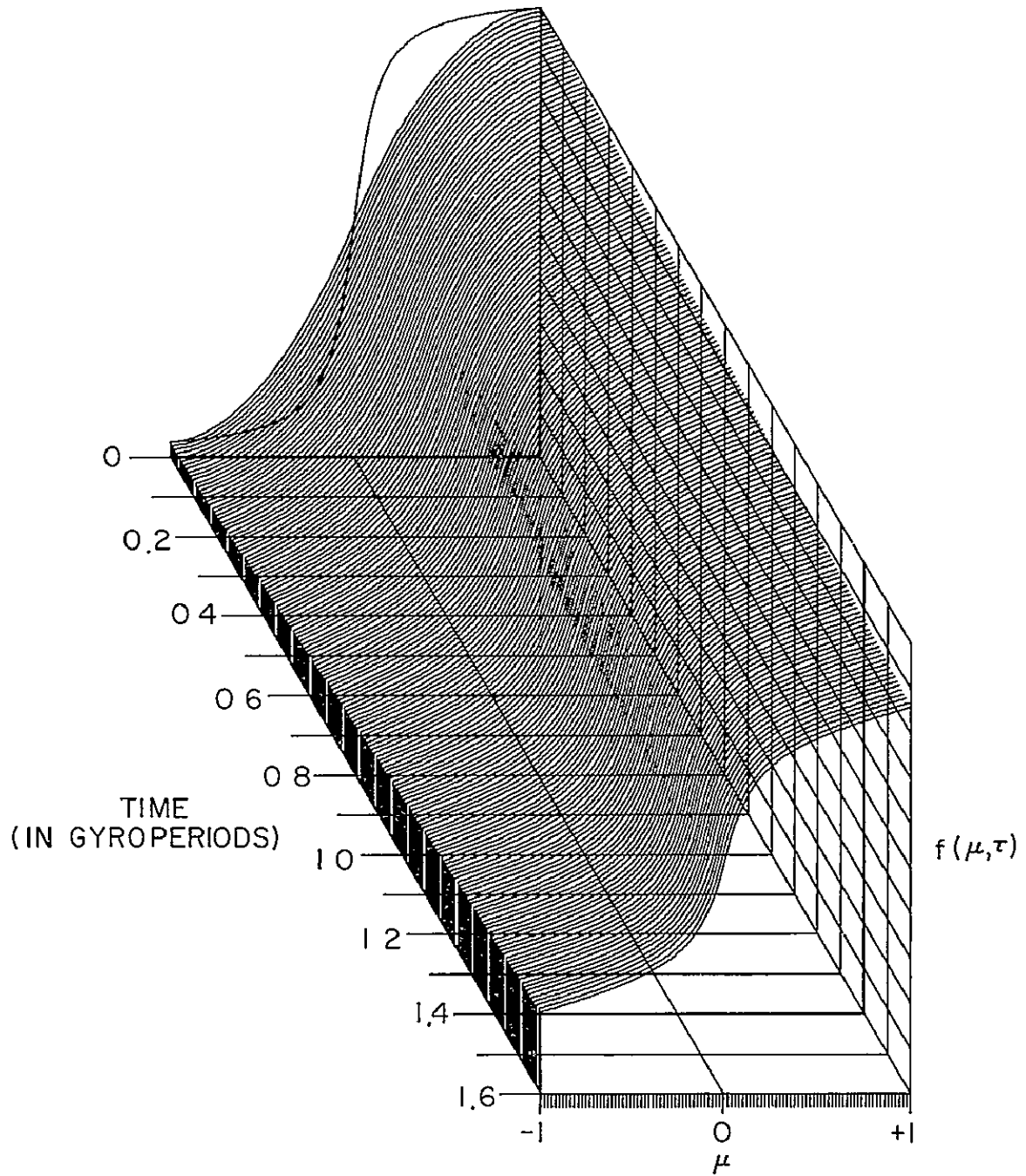
Figure 18. The first local approximation to the quasi-linear gyro-phase averaged probability distribution function for the evolution of a broad beam injected parallel to the mean magnetic field. Compare with figure 5 of paper I.
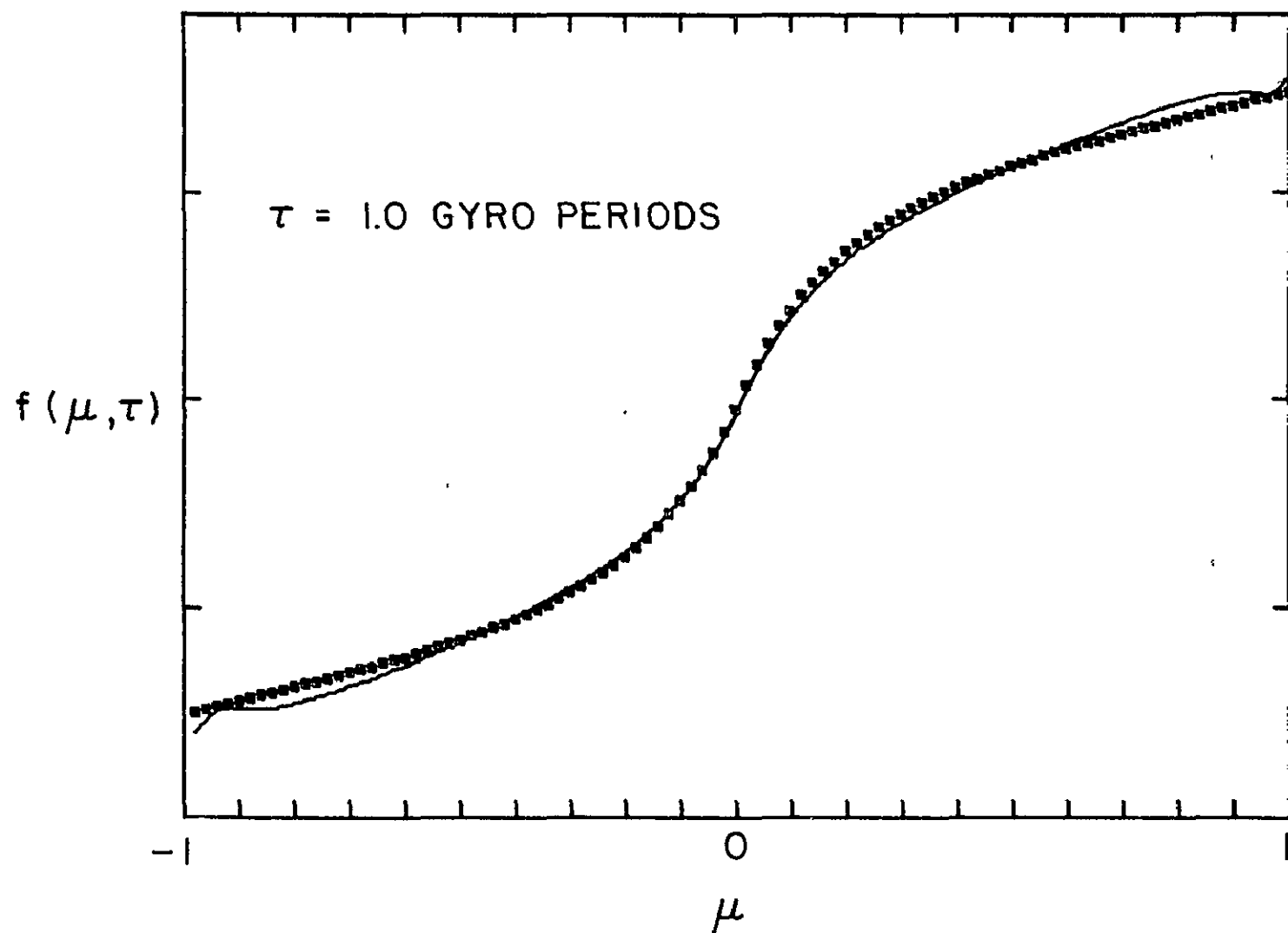
Figure 19. A direct comparison, at one Larmor period, of the quasi-linear gyro-phase averaged probability distribution function with its first local approximation for the broad parallel beam injection of figure 18. The squares give the first local approximation.
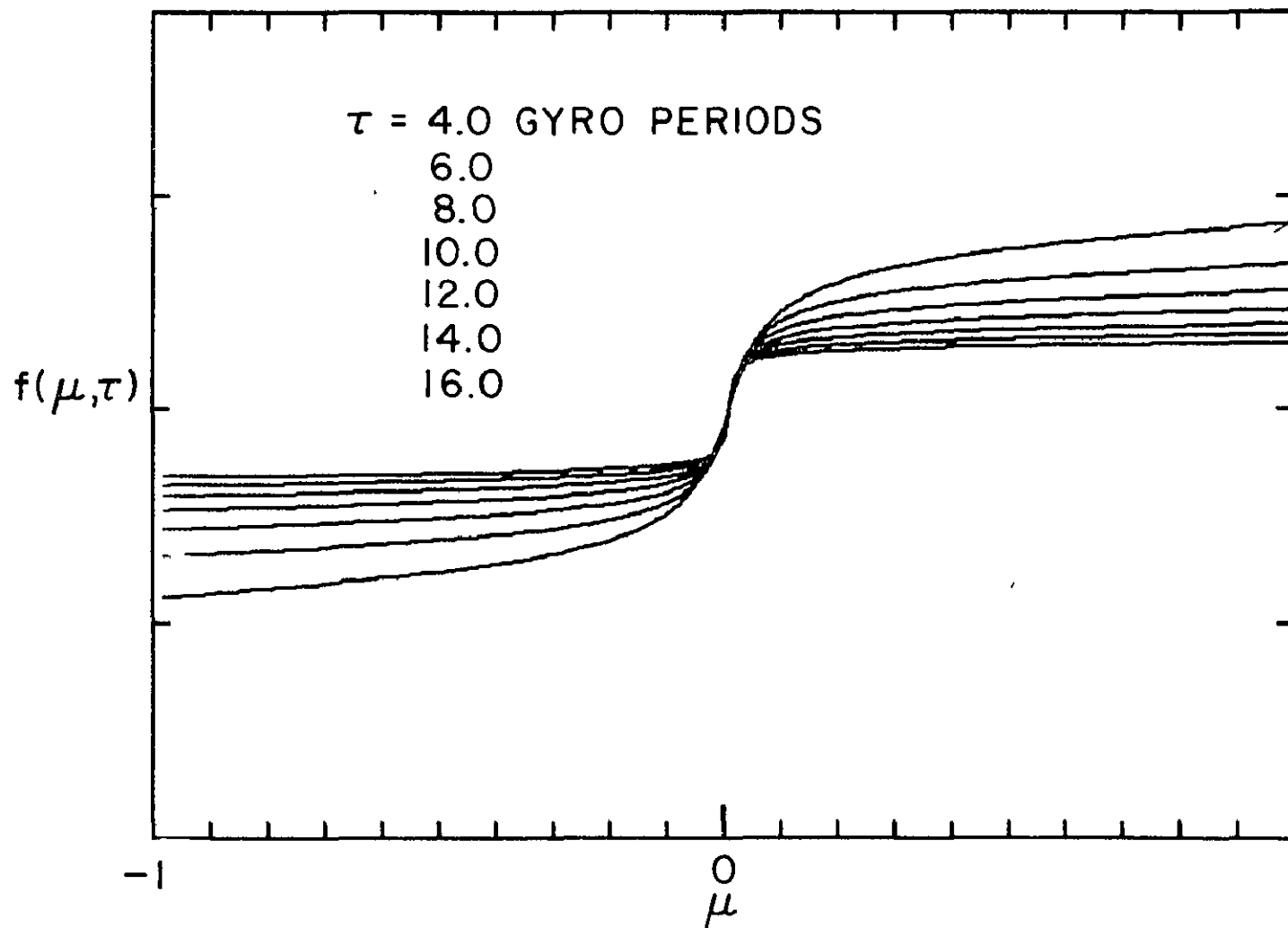
Figure 20. The first local approximation to the quasi-linear gyro-phase averaged probability distribution function for the broad parallel beam injection of figure 18. The curves give the probability distribution function from four to sixteen Larmor periods in intervals of two Larmor periods.

In general, it seems that we can conclude that the local approximation to

the quasi-linear diabatic probability distribution function is very accurate after

an elapsed time of about one Larmor period during which rapid wave-like

transients decay away in the diabatic solution. The local approximation is

reinitialized so that after the decay of the transients the diabatic solution and

its local approximation join together and then evolve in unison through the

longest times . . . several Larmor periods in some cases . . . for which our

numerical solutions have been obtained. Of course, these general conclusions

must be qualified by the fact that we have not even attempted a thorough in-

vestigation of the available parameter space; we have studied a single particle

energy for which the particle Larmor radius is equal to the correlation length

in the turbulent magnetic field, and we have considered only a single value for

the parameter, η, which measures the relative strengths of the random and

mean fields. Furthermore, we have not addressed the question in this paper

of what happens to any of these solutions for times much larger than those

available to us in our numerical results. In the next paper in this series we will

develop strong evidence that the local approximation remains faithful to the

diabatic solution for all post transient times.

We have only mentioned briefly in paper I. the many precautions which we

have taken to ensure the accuracy of the numerical results presented in that

paper and also in this one. We feel that the most convincing argument that we

can make for this accuracy is based on the direct comparison figures which

we have presented in this section. The completely explicit marching routine

which we used to obtain the quasi-linear diabatic solutions in paper I. is far

removed from the explicit-implicit routine which we used to obtain the local

approximations presented in this paper. If either of these integration routines

were to develop significant errors, it would be highly unlikely that the integration

results would compare as favorably as they do for such a variety of initial

conditions. On this basis, our confidence in the accuracy of both integration

routines remains high.

V. DISCUSSION

We have developed a new approximation method which we call the local

approximation method. We believe that this new method is applicable to a wide

class of equations of importance in many kinetic and transport theories. In

the local approximation method an expansion for small values of the Laplace

variable is carried out in order to obtain the slow evolution of the mechanical

system under consideration. This procedure is in full accord with the aims of

standard kinetic theory expansions which rely, however, on the existence of a

small parameter (ϵ = ratio of two time scales) to obtain the evolution on the

slow time scale. The standard approach yields asymptotic expansions in the

small parameter which often become inadequate when non-uniform behavior

occurs in the phase space associated with the mechanical system. For the test

particle propagation problem under consideration here a non-uniformity occurs

in the vicinity of $\mu = 0$ due to mixing of the two time scales. Our new technique successfully removes this non-uniformity.

We have applied the local approximation method to two examples in order to demonstrate its use in detail. The first of these was a simple exactly soluble problem all of whose local approximations were available as explicit analytic expressions. The second example was the quasi-linear diabatic equation for test particle propagation in an axi-symmetric slab model of magnetostatic turbulence which was derived from first principles in paper I.. For this example the local approximations to the probability distribution function are available only numerically; we have presented only the first local approximation in this paper. For both examples the available local approximations generally are remarkably successful, but, there is one exception.

The conditions under which the local approximations either succeed or fail remain essentially unknown to us. In fact, we have not formulated a quantitative criterion for the success or failure of any of the local approximations. For the two examples presented in this paper we find the local approximations generally successful based on visual inspections of plots of the local approximations compared to plots of the exact solutions. These are not asymptotic approximations which we have constructed here, and the usual tests for asymptoticity cannot be applied. Often we find that the difference between the exact and approximate solutions is large, but in these cases the approximate solutions

nevertheless appear to give an accurate representation of the average behavior of the exact solutions. Even more often we find the approximate solutions fit the exact solutions extremely well. Based on the results of these two examples, and because this method is so unique, we feel that a much deeper investigation of the local approximation method is justified.

The particular construction which we have presented in this paper is probably part of a more general approach. We have quasi-uniformized the Maclaurin series expansion of the Laplace transform of the kernel through the use of some of the available Padé approximants. Other of the available Padé approximants may be useful. We imagine also that there are other uniformization techniques which could be used, and which would result in other approximation methods. The exact solutions of the two examples which we have presented here are clearly dominated for large times by contributions to their inverse Laplace transforms from the vicinity of the Laplace space origin. Thus, the approximations to the Laplace transforms of the relevant kernels which we have used, which are most accurate in the vicinity of the origin, are appropriate. In other applications, various types of uniformization techniques might be applied to Taylor series expansions about other points on the complex Laplace plane. It would seem to us that the key to this method is to introduce a simplification of the exact solution under consideration which is most accurate in a domain in the complex Laplace plane that has been determined, through

some other means, to dominate the inverse Laplace transformation. The use
of Padé approximants for this purpose is particularly attractive since the
result of this simplification is likely to be explicitly inverse Laplace trans-
formable, and the failure of the simplification far away from the point of
domination is not likely to be catastrophic.

Based on visual inspection of the appropriate curves for our exactly soluble
example, we would say that the first local approximation remains viable for all
positive time. From our construction of the first local approximation to the
low order Laplace modes, we conclude that the approximate eigenfunctions
and eigenvalues accurately represent their exact counterparts no matter how
small the Laplace variable. These conclusions indicate that the first local
approximation to the quasi-linear diabatic solutions presented in paper I. may
remain accurate beyond the ends of the available numerical evaluations. In
the next paper in this series we investigate this issue more fully and conclude
that this indication is correct.

ACKNOWLEDGEMENT

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APPENDIX A

THE RELATIONSHIP BETWEEN EQUATION II.1 AND
OTHER KINETIC THEORIES

Consider the pair of coupled equations,

$$\frac{\partial F}{\partial \tau} + H_0 F = \epsilon \, L \, G \qquad\qquad A.1$$

and,

$$\frac{\partial G}{\partial \tau} + H \, G = I \, F \qquad\qquad A.2$$

in which,

$$L = AI \qquad\qquad A.3$$

where A, H_0, H, and I are linear time independent operators which operate on everything to their right. The generalization in which H and I are time dependent operators will be considered later in this appendix.

Equation A.2 can be solved formally; we find,

$$G(\tau) = U(\tau)\, G(0) + \int_0^\tau d\lambda \, U(\lambda)\, IF(\tau - \lambda) \qquad\qquad A.4$$

in which,

$$U(\tau) = e^{-H\tau} \qquad\qquad A.5$$

Upon substituting equation A.4 into equation A.1, we find,

$$\frac{\partial F}{\partial \tau} + H_0 F = \epsilon LU(\tau) \, G(0) + \epsilon \int_0^\tau d\lambda \, K(\lambda) \, F(\tau - \lambda) \qquad \text{A.6}$$

in which,

$$K(\lambda) = AI \, e^{-H\lambda} \, I \qquad \text{A.7}$$

In the following, we will show that equation A.6 is equivalent to equation II.1 for the purpose of applying the local approximation method. Then we will demonstrate how all of the fundamental approaches to statistical mechanics can be put into the structure of equations A.1 and A.2 with, in some cases, time dependent operators. Those cases with time dependent operators will be considered last.

In situations where the operator, H_0, appears it is the Hamiltonian or Liouville operator which generates the effects of an external or averaged force field. In paper I. equations I.8 and I.10 contain such a term, but it does not appear later in that paper because of the gyro-phase averaging done in section III. Thus, the effects of an external or averaged force field have not been included in equation II.1 through H_0 .

The quantity G(0) plays the role of fluctuations or correlations in the initial conditions; it is traditional to ignore it. In the few cases that G(0) has not been set initially to zero it has been found that it has negligible effects beyond the initial transients in the initial value solutions which are available.[7] We feel that

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the neglect of G(0) in situations where a non-Markovian theory is necessary

has not been justified, however, for the purpose of this study we have conformed

to tradition.

It is important to realize that the inclusion of the two terms discussed in

the previous two paragraphs does not present an obstacle for the construction

of the local approximation method given in the main text of this paper. In the

local approximation method the effects of the integral term in equation A.6 are

approximated. If these two additional terms were included, they would simply

be carried along through the calculation. Thus, for the purpose of constructing

the local approximation method we consider equation A.6 completely equivalent

to equation II.1, and therefore, consider equations A.1 and A.2 also completely

equivalent.

In the following we give a list of all the fundamental approaches which are

basic to the construction of kinetic theory. In each sub-section we give the

definitions and changes of notation which are necessary to put each fundamental

approach into the form of equations A.1 and A.2. These listings are brief, but,

the references necessary for further understanding are included.

a). The Master equation formalism[7,9]

We consider M particles in a phase space $\{x_i, v_i\}$ where i equals 1

through M. The quantity, F^M, is the Liouville probability distribution function

for the M particles which obeys the Liouville equation. The averaging operator,

A, is given by,

$$A = \int \prod_{i=1}^{M} d^3x_i \qquad\qquad \text{A.8}$$

The Master equation-gives the evolution of,

$$F = AF^M \qquad\qquad \text{A.9}$$

It follows from the pair of coupled equations, equations A.1 and A.2, in which,

$$G = (1 - A) F^M \qquad\qquad \text{A.10}$$

$$H = \sum_{i=1}^{M} \mathbf{v}_i \cdot \frac{\partial}{\partial x_i} - I(1 - A) \qquad\qquad \text{A.11}$$

and, for particles which interact through a two-body central potential,

$$I = \sum_{\substack{i < j \\ i,j = 1}}^{M} \frac{\partial \phi_{ij}}{\partial x_i} \cdot \left(\frac{\partial}{\partial \mathbf{v}_i} - \frac{\partial}{\partial \mathbf{v}_j} \right) \qquad\qquad \text{A.12}$$

The effects of an external force field could be included in H_0.

b). The BBGKY Hierarchy[7,10]

First consider the case in which three-body interactions are neglected. Then the one and two-body probability distribution functions become decoupled from the remainder of the hierarchy of cluster distributions. Equations A.1 and A.2 give the evolution of the one-particle probability distribution function, F^1, and the two-particle probability distribution function, F^2, after the following substitutions; let the averaging operator be,

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$$A = \int d^3 x_2 \, d^3 v_2 \qquad\qquad A.13$$

and let $F = F^1$ and $G = F^2 - F^1 F^1$. Then F and G obey equations A.1 and A.2 with

$$H = v_1 \cdot \frac{\partial}{\partial x_1} + v_2 \cdot \frac{\partial}{\partial x_2} - I \qquad\qquad A.14$$

and, again for a two body central potential interaction,

$$I = \frac{\partial \phi_{12}}{\partial x_1} \left(\frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_2} \right) \qquad\qquad A.15$$

In this case, even in the absence of an external force field, there is an H_0 given by

$$H_0 = K = v_1 \cdot \frac{\partial}{\partial x_1} \qquad\qquad A.16$$

In the absence of a truncation of the BBGKY hierarchy as in the preceding paragraph, we can introduce a treatment of the entire hierarchy which is similar to that given in the preceding section in the Master equation formalism. The s-particle probability distribution function obeys the hierarchy member equation

$$\frac{\partial F^s}{\partial \tau} + K_s F^s - I_s F^s = L_s F^{s+1} \qquad\qquad A.17$$

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with the operators K_s, I_s, and L_s similar to the K, I, and L operators (L = AI) given in the preceding paragraph, but with summations over the appropriate numbers of particles. With the introduction of the M'th order vector, F, given by,

$$F = \begin{pmatrix} F^1 \\ F^2 \\ F^3 \\ \cdot \\ \end{pmatrix} \qquad\qquad A.18$$

the entire hierarchy of equations can be represented as a matrix equation,

$$\frac{\partial F}{\partial \tau} + KF - IF = LF \qquad\qquad A.19$$

where, K, I, and L are the M'th rank matrices whose elements are all zeros except for the following:

$$K_{11} = K_1 \qquad I_{11} = I_1 \qquad L_{1,j+1} = L_1 \qquad\qquad A.20$$

Now, a further averaging, A', of equation A.19 which parallels that used to produce the Master equation from the Liouville equation in the preceding section could be formulated and would lead to a pair of coupled equations, equations A.1 and A.2, for A' F and (1 − A')F.

c). The Klimontovich formalism[12,13,14,15]

We illustrate the Klimontovich formalism through its application to the Lorentz gas. In this application the operators, H and I, which appear are time

independent. In a more general application this formalism contains time dependent operators, but then, it can be treated as in the last examples in this appendix.

We consider the particle micro-density, N, for M ions and one electron, which obeys the Liouville equation,

$$\frac{\partial N}{\partial \tau} + \mathbf{v} \cdot \frac{\partial N}{\partial \mathbf{x}} - \left(\frac{\partial \Phi}{\partial \mathbf{x}}\right) \cdot \frac{\partial N}{\partial \mathbf{v}} = 0 \qquad \text{A.21}$$

in which,

$$\Phi = \sum_{\imath = 1}^{M} \phi(\mathbf{x} - \mathbf{x}_\imath) \qquad \text{A.22}$$

and ϕ represents the two body potential interaction. The averaging operator in this case is given by,

$$A = \left(\frac{1}{V}\right)^{M+1} \int d^3 v_e d^3 x_e \prod_{\imath = 1}^{M} d^3 x_\imath F^{M+1} \qquad \text{A.23}$$

in which F^{M+1} is the coarse grained Liouville probability distribution function for the M ions and the one electron. Now equations A.1 and A.2 give the evolution of $F = AN$ and $G = (1 - A)N$ with

$$I = \frac{\partial \Phi}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{v}} \qquad \text{A.24}$$

and,

$$H = v \cdot \frac{\partial}{\partial x} - I \qquad\qquad A.25$$

We note in passing that this formalism is equivalent to the multiple scattering formalism.[13]

d). Test particle propagation in magnetostatic turbulence with no approximations.

This series of papers is concerned with the quasi-linear approximation to the ensemble averaged probability distribution function for test particle propagation in magnetostatic turbulence. We have shown that the quasi-linear diabatic equation which governs the probability distribution function within this framework is in the form of equation II.1. It is also true that the equations which govern the exact ensemble averaged probability distribution function can be put in the form of the coupled pair of equations given by equations A.1 and A.2. Thus, the local approximation method can be applied to an exact formulation of the test particle propagation problem as well as the quasi-linear formulation.

Equations 13 and 14 of Klimas and Sandri[16] can be rewritten as equations A.1 and A.2 through the following changes in notation: Let $F = f$, and $G = f'/\epsilon'$. Then with ensemble averaging for the A operator, and,

$$H_0 = K + \epsilon \mathcal{L} \qquad\qquad A.26$$

$$I = - \mathcal{L}' \qquad\qquad A.27$$

and

$$H = K + \epsilon \mathcal{L} + \epsilon' (1 - A) \mathcal{L}' \qquad \text{A.28}$$

the equations due to Klimas and Sandri become identical to equations A.1 and A.2 with $(\epsilon')^2$ in place of ϵ.

In the event that the H and I operators of equation A.2 are functions of time, the construction of a closed equation for F is still possible, and the local approximation method can still be applied. Here we establish the groundwork for this situation, and then discuss several examples in which it occurs.

With time dependent H and I, the formal solution of equation A.2 is given by,

$$G(\tau) = U(\tau) \, G(0) + U(\tau) \int_0^\tau d\lambda \, U^{-1}(\lambda) \, I(\lambda) \, F(\lambda) \qquad \text{A.29}$$

in which,

$$U(\tau) = T \left[\exp \left(- \int_0^\tau ds \, H(s) \right) \right] \qquad \text{A.30}$$

where T is the time ordering or chronological operator.[17,18,19] The closed equation for F is now given by,

$$\frac{\partial F}{\partial \tau} + H_0 F = \epsilon \, L \, U(\tau) \, G(0) + \epsilon \int_0^\tau d\lambda \, K(\tau, \lambda) \, F(\lambda) \qquad \text{A.31}$$

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in which,

$$K(\tau, \lambda) = AI(\tau) U(\tau) U^{-1}(\lambda) I(\lambda) \qquad \text{A.32}$$

Although we have lost the-convolution form for the integral term in this equation, the Laplace transform can still be constructed. Rather than an algebraic equation in the Laplace variable, p, we now obtain an integral equation in this variable;

$$p\tilde{F}(p) - F(0) + H_0\tilde{F}(p) = \left(\frac{\epsilon}{2\pi i}\right) A \int_{-i\infty}^{i\infty} dp' \; \tilde{I}(p') \tilde{U}(p - p') G(0)$$

$$\text{A.33}$$

$$+ \left(\frac{\epsilon}{2\pi i}\right) A \int_{-i\infty}^{i\infty} d\omega \; \tilde{K}(p, \omega) \tilde{F}(\omega)$$

where,

$$\tilde{K}(p, \omega) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dp' \; \tilde{\Im}(p') \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dp'' \; \tilde{\Im}^+(p'') \left(\frac{1}{p' - p}\right) \left(\frac{1}{p'' + \omega + p' - p}\right) \text{A.34}$$

in which $\tilde{\Im}(p)$ and $\tilde{\Im}^+(p)$ are the Laplace transforms of,

$$\Im(\tau) = I(\tau) U(\tau) \qquad \Im^+(\tau) = U^{-1}(\tau) I(\tau) \qquad \text{A.35}$$

Of course this case is considerably more complicated than that discussed above in which H and I are time independent. Nevertheless, we are in a position to apply the local approximation method at this point by producing a continued fraction expansion of $\tilde{K}(p, \omega)$ about the origin in p.

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In the following we list a few important approaches which contain time dependent operators, but, by the reasoning above are amenable to the local approximation method.

e). The Klimontovich formalism[12, 13, 14, 15]

When the Klimontovich formalism is applied to a more general problem than the Lorentz gas, then the quantity, Φ, given by equation A.22 becomes a function of time through the time dependent positions of all of the particles. Otherwise, the notation given in section c). is left essentially unchanged by this generalization, and thus, equations A.1 and A.2 follow. In this case, however, both H and I are time dependent operators.

f). The passive additive theory of Obukhov[13, 14]

This is a macro-turbulence theory which gives the evolution of C, the concentration of a passive additive. The evolution of C is given by,

$$\frac{\partial C}{\partial \tau} + V \cdot \frac{\partial C}{\partial x} = D \nabla^2 C \qquad\qquad A.36$$

in which D is a diffusion coefficient, and V is the centered random field distributed over an ensemble; it is generally a function of time. This equation leads to the pair of coupled equations A.1 and A.2 with A an ensemble averaging operator, $F = AC$, $G = (1 - A)C$,

$$I = - [| (1 - A) V |] \cdot \frac{\partial}{\partial x} \qquad\qquad A.37$$

$$H = (1 - A) \, \mathbf{V} \cdot \frac{\partial}{\partial \mathbf{x}} - D\nabla^2 \qquad \text{A.38}$$

and,

$$H_0 = A\mathbf{V} \cdot \frac{\partial}{\partial \mathbf{x}} - D\nabla^2 A \qquad \text{A.39}$$

The symbol, $[| \quad |]$, is used to indicate that the averaging operator does not operate on anything to the right of the symbol.

g). Evolution of a system with a random Hamiltonian

Equations 6 and 8 of Kaufman are identical to equations A.1 and A.2 if the following identifications are made: Let $F = P$, $G = \delta\rho$, and A the averaging operation given by Kaufman. Furthermore, let

$$H_0 = \langle L \rangle \qquad \text{A.40}$$

$$I = - \delta L \qquad \text{A.41}$$

and,

$$H = H_0 + (1 - A) \, \delta L \qquad \text{A.42}$$

In this approach, δL is considered a function of time. Thus, this approach could be applied to the problem of particle propagation in electromagnetic turbulence. The approach of Klimas and Sandri was taken from this one, but their approach is limited to magnetostatic turbulence due to their assumption

of time independent operators. Kaufman's work provides the generalization

necessary to extend the test particle analysis to time dependent electromagnetic

turbulence. The local approximation method will still play a role in that ex

tension of the present work.

APPENDIX B

FAILURE OF DEGENERATE PERTURBATION ANALYSIS
WITH ADIABATIC EIGENBASIS

The equation which generates the Sturm-Liouville eigenfunctions can be

written (see equation I.54, paper I.)

$$(D + \lambda_{m,\ell})| \; m, \; \ell \rangle = 0 \qquad \text{A.1}$$

with,

$$D \equiv \frac{\partial}{\partial \mu} \; (1 - \mu^2) \; \tilde{\mathcal{H}}(\mu, \; p) \; \frac{\partial}{\partial \mu} \qquad \text{B.2}$$

In this appendix, we adopt an indexing system for the eigenfunctions, and eigen-

values, which differs from that used in the main text. We use the index, m, to

denote each pair of even and odd eigenfunctions, and the index, ℓ, to denote which

of each pair is being considered; the even eigenfunctions will be labeled $\ell = 0$,

and the odd eigenfunctions, $\ell = 1$. We attempt a perturbation expansion of the

eigenfunctions, and eigenvalues, in powers of the Laplace variable, p, which we

assume is small. Thus,

$$\lambda_{m,\ell} = \lambda_m^{(0)} + p\lambda_{m,\ell}^{(1)} + \cdots \qquad \text{B.3}$$

and

$$|m, \; \ell \rangle = |m, \; \ell \rangle^{(0)} + p|m, \; \ell \rangle^{(1)} + \cdots \qquad \text{B.4}$$

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The zeroth order quantities in these expansions are the adiabatic approximations. We have found that these adiabatic eigenfunctions are doubly degenerate with a pair of even and odd eigenfunctions for each eigenvalue. Thus, the zeroth order approximation to the eigenvalues does not depend on the index, ℓ. We also expand the differential operator, D, in powers of p according to

$$D = D^{(0)} + pD^{(1)} + \cdots \qquad \text{B.5}$$

in which

$$D^{(n)} = (-1)^n \frac{\partial}{\partial \mu} (1 - \mu^2) \, \mathbb{M}_n \, \frac{\partial}{\partial \mu} \qquad \text{B.6}$$

and the $\mathbb{M}_n$ are the n'th moments of $\tilde{\mathcal{K}}(\mu, p)$. On equating coefficients of increasing powers of p, we find, to zero'th order in p,

$$D^{(0)} \, |m, \, \ell\rangle^{(0)} + \lambda_m^{(0)} \, |m, \, \ell\rangle^{(0)} = 0 \qquad \text{B.7}$$

and, to first order in p,

$$(D^{(0)} + \lambda_m^{(0)}) \, |m, \, \ell\rangle^{(1)} = - (D^{(1)} + \lambda_{m,\ell}^{(1)}) \, |m, \, \ell\rangle^{(0)} \qquad \text{B.8}$$

We expand $|\, m, \, \ell^{\,(1)}\rangle$, using the adiabatic eigenfunctions as a basis,

$$|m, \, \ell\rangle^{(1)} = |m', \, \ell\rangle^{(0)} \; {}^{(0)}\langle m', \, \ell'|m, \, \ell\rangle^{(1)} \qquad \text{B.9}$$

and substitute this expression in to equation B.8 to obtain

$$(\lambda_m^{(0)} - \lambda_m^{(0)}) \, |m', \, \ell\rangle^{(0)} \; {}^{(0)}\langle m', \, \ell'|\, m, \, \ell\rangle^{(1)}$$

$$\text{B.10}$$

$$= - (D^{(1)} + \lambda_{m,\ell}^{(1)}) \, |m, \, \ell\rangle^{(0)}$$

From this expression, using the orthogonality of the adiabatic eigenfunctions, we find,

$$^{(0)}\langle n, k, |m, \ell\rangle^{(1)} = -\left(\frac{1}{\lambda_m^{(0)} - \lambda_n^{(0)}}\right)[\lambda_{m,\ell}^{(1)}\delta_{m,n}\delta_{\ell,k} + \, ^{(0)}\langle n, k|D^{(1)}|m, \ell\rangle^{(0)}] \qquad \text{B.11}$$

where

$$^{(0)}\langle n, k|D^{(1)}|m, \ell\rangle^{(0)} = -\int_{-1}^{1} d\mu \, \psi_{n,k}^{(0)} \left(\frac{\partial}{\partial\mu}(1 - \mu^2)\, m_1 \frac{\partial}{\partial\mu}\right)\psi_{m,\ell}^{(0)}$$

$$\text{B.12}$$

$$= \int_{-1}^{1} d\mu \, \frac{\partial\psi_{n,k}^{(0)}}{\partial\mu}(1 - \mu^2)\, m_1 \frac{\partial\psi_{m,\ell}^{(0)}}{\partial\mu}$$

by integration by parts. Notice that equation B.12 is given by zero unless $\ell = p$. In order to prevent the divergence of equation B.11 when m = n, we set,

$$\lambda_{m,\ell}^{(1)} + \, ^{(0)}\langle n, k|D^{(1)}|m, \ell\rangle^{(0)} = 0 \qquad \text{B.13}$$

and then,

$$^{(0)}\langle n, k|m, \ell\rangle^{(1)} = -\frac{\delta_{k,\ell}}{\lambda_m^{(0)} - \lambda_n^{(0)}} \, ^{(0)}\langle n, \ell|D^{(1)}|m, \ell\rangle^{(0)} \qquad (m \neq n) \qquad \text{B.14}$$

$$= 0 \qquad\qquad\qquad\qquad\qquad\qquad (m = n)$$

both of which provide the first order corrections to the adiabatic eigenfunctions and eigenvalues. However, from equation B.12 we can see that when $\ell = 1$, $^{(0)}\langle n, 1 |D^{(1)}| m, 1\rangle^{(0)}$ is given by a divergent expression due to the discontinuous

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behavior of the odd adiabatic eigenfunctions at $\mu = 0$. Thus, in this expansion scheme, we find that the first order corrections to the eigenfunctions and eigenvalues are misordered due to the discontinuous nature of the odd adiabatic eigenfunctions which are used as a basis for the expansion.

The conclusions made in the previous paragraph can only be made if we show in addition that the degenerate eigenfunctions of definite parity in μ are the appropriate linear combination within the degenerate space (p = 0) which evolve continuously as p grows from p = 0. This is ordinarily done by showing that the off diagonal terms of the secular matrix are zero, viz.,

$$^{(0)}\langle n, \ell \,|\, p\, D^{(1)} \,|\, n, m \rangle^{(0)} = 0 \quad m \neq \ell \qquad \text{B.15}$$

As argued below equation B.12 these integrals do vanish identically because $D^{(1)}$ is even in μ. From another point of view we have graphically shown in paper I. (figures 6 and 7) that as $p \to 0$ the eigenfunctions represented by $|n, \ell\rangle^{(0)}$ of definite parity are the limiting form of the exact eigenfunctions.

APPENDIX C

THE n'TH LOCAL KERNEL FOR n = ZERO THROUGH THREE

The $\bar{K}_n(\lambda)$ which appear in equation $\overline{\overline{\text{II}}}.13$ can be obtained by inverse Laplace transforming the convergents which appear in the boxes with dark outlines in the operator Pade' table shown in figure 1. From the Pade' table,

$$\tilde{K}_0(p) = M_0 \qquad\qquad C.1$$

Therefore,

$$K_0(\tau) = M_0 \delta(\tau) \qquad\qquad C.2$$

Similarly,

$$\tilde{K}_1(p) = M_0 - M_1 P \qquad\qquad C.3$$

and therefore,

$$K_1(\tau) = M_0 \delta(\tau) - M_1 \dot{\delta}(\tau) \qquad\qquad C.4$$

At this point it appears as though we are producing truncations of a one dimensional multipole expansion of the kernel with the order of the truncation increasing with the level of the local approximation. This impression would be correct if we were to continue down the M = 0 column on the Padé table. However, to obtain the second local approximation we now leave that column and proceed down the "staircase" of convergents. From this point on in our procedure we have found no simple characterization of the resulting $K_n(\tau)$.

The second convergent is given from the Pade' table by,

$$\tilde{K}_2(p) = \frac{M_0 + \left[\frac{1}{2} M_0 M_1^{-1} M_2 - M_1\right] p}{1 + \frac{1}{2} M_1^{-1} M_2 p}$$

C.5

For large p we have,

$$\tilde{K}_2(p) \underset{\uparrow}{\tilde{p}} \tilde{K}_2(\infty)$$

C.6

in which,

$$\tilde{K}_2(\infty) = \frac{\frac{1}{2} M_0 M_1^{-1} M_2 - M_1}{\frac{1}{2} M_1^{-1} M_2}$$

C.7

To obtain the inverse Laplace transform of $\tilde{K}_2(p)$ we use,

$$\tilde{K}_2(p) = [\tilde{K}_2(p) - \tilde{K}_2(\infty)] + \tilde{K}_2(\infty)$$

C.8

$$= \frac{4(M_1 M_2^{-1})^2 M_1}{p + 2M_2^{-1} M_1} + (M_0 - 2M_1 M_2^{-1} M_1)$$

With this form for $\tilde{K}_2(p)$ find,

$$K_2(\tau) = (M_0 - 2M_1 M_2^{-1} M_1)\, \delta(\tau) + 4(M_1 M_2^{-1})^2 M_1\, e^{-2M_2^{-1} M_1 \tau}$$

C.9

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Similarly,

$$K_3(\tau) = \left[M_0 - \frac{9}{2} (M_2 M_3^{-1})^2 M_2 \right] \delta(\tau)$$

$$- \left[M_1 - \frac{3}{2} M_2 M_3^{-1} M_2 \right] \dot{\delta}(\tau) \qquad \text{B.10}$$

$$+ \quad \frac{27}{2} (M_2 M_3^{-1})^3 M_2 \; e^{-2 M_3^{-1} M_2 \tau}$$

We have not computed any of the higher order $K_n(\tau)$.